# Assortment of combinatorial theorems

## Discrete Mathematics, 5th lecture

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# A number of slightly similar theorems

- Given a structure (graph, matrix, p.o. set), we define two kinds of objects on it:
- 1st kind — they connect something.
- 2nd kind — they separate something, or cover something.
- Theorems state that the size of the smallest objects of the second kind is equal to the number of "mutually independent" objects of the first kind.
- The proofs consist of two parts, "$\leqslant$" and "$\geqslant$".
  - One part is usually much easier than the other one.
  - The following lecture slides contain only proofs (sketches) for the hard directions.

# Recall: Flows and cuts

- Network — directed graph $G = (V, E)$ with a source $s \in V$ and a sink $t \in V$, and a mapping $\psi : E \to \mathbb{R}_+$.
- Flow on $(G, \psi)$ — mapping $\varphi : E \to \mathbb{R}_+$, such that
  - $\forall e \in E : \varphi(e) \leqslant \psi(e)$;
  - $\forall v \in V \backslash \{s, t\}$: $\overrightarrow{\deg}_\varphi(v) = \overleftarrow{\deg}_\varphi(v)$.
- Value of flow $\varphi$ is equal to $\overleftarrow{\deg}_\varphi(s)$ or $\overrightarrow{\deg}_\varphi(t)$.
- Cut in $(G, \psi)$ — a set of edges $E' \subseteq E$, such that all paths from $s$ to $t$ use some edge in $E'$.
- Value of cut $E'$ is $\sum_{e \in E'} \psi(e)$.

**Theorem (Ford and Fulkerson, 1962)**

*In a network, maximum value of flows and minimum value of cuts are equal.*

# Term ranks and covers of matrices

Let $M$ be a $m \times n$ matrix with 0/1 entries.

- A line of a matrix is its row or column.
- A partial transversal is a selection of entries "1" in $M$, such that no two of them lie on the same line.
- The term rank of $M$ is the maximum size of its partial transversals.
- A cover of $M$ is a set of its lines that contain all 1-s in $M$.

**Theorem (König-Egerváry, 1931)**

*The term rank of a 0/1-matrix M is equal to the size of its minimum covers.*

# Proof of König-Egerváry's theorem

- Easy direction: minimum cover must be at least the term rank. In the other direction...
- Let $c$ be the term rank of $M$.
- Define a network as follows:
  - Vertices: $x_1, \ldots, x_m$, $y_1, \ldots, y_n$,$s$,$t$.
  - Edges: $s \xrightarrow{1} x_i$; $y_j \xrightarrow{1} t$; $x_i \xrightarrow{c+1} y_j$ (if $M_{ij} = 1$)
- In a maximal flow $\varphi$ we have $\forall e : \varphi(e) \in \{0, 1\}$.
- The value of $\varphi$ is $c$.
- In a corresponding minimum cut, there are edges $s \xrightarrow{1} x_{i_1}, \ldots, s \xrightarrow{1} x_{i_k}$, $y_{j_1} \xrightarrow{1} t, \ldots, y_{j_l} \xrightarrow{1} t$, where $k + l = c$.
- These select $k$ rows and $l$ columns that cover $M$.

# Matchings and coverings

Let $G = (V, E)$ be a simple graph.

**Definition**

A matching (*kooskõla*) is a set $M \subseteq E$, such that $\forall v \in V : \deg_M(v) \leqslant 1$.

**Definition**

A covering (*kate*) is a set $V \subseteq V$, such that each edge in $E$ has at least one end-point in $C$.

# Theorems of Hall and König

For $S \subseteq V$, let $N(S) \subseteq V$ denote the neighbourhood of $S$ — the set of vertices that neighbour at least one vertex in $S$.
Let maximum matching / minimum cover denote the matching(s) / cover(s) with maximum/minimum cardinality.

**Theorem (Hall's Marriage theorem, 1935)**

*Let $G = (X \cup Y, E)$ be a bipartite graph with parts $X$ and $Y$. $G$ has a matching $M$ with $\forall x \in X : \deg_M(x) = 1$ iff $\forall S \subseteq X : |N(S)| \geqslant |S|$.*

**Theorem (König's theorem for matrices, 1931)**

*Let $G = (X \cup Y, E)$ be a bipartite graph. Maximum matchings and minimum covers in $G$ have the same cardinality.*

# Proof of König's theorem

- Given bipartite graph $G = (X \cup Y, E)$, consider the adjacency matrix of $G$.
  - Size: $|X| \times |Y|$, rows indexed by $X$, columns indexed by $Y$.
  - Entry in position $(u, v)$ equals 1 iff $(u, v) \in E$, otherwise it equals 0.
- In this matrix
  - Partial transversals correspond to matchings in $G$;
  - Covers correspond to coverings in $G$.

# Proof of Hall's theorem

- Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall's criterion.
- Let $C$ be a minimal covering for $G$.
- If $C$ is a cover then $N(X \backslash C) \subseteq Y \cap C$.
- Thus $|Y \cap C| \geqslant |N(X \backslash C)| \geqslant |X \backslash C|$.
- Thus $|C| = |X \cap C| + |Y \cap C| \geqslant |X \cap C| + |X \backslash C| = |X|$.
- Thus there is a matching of size at least $|X|$.

# Separating sets and disjoint paths

## Definition

Let $G = (V, E)$ be a connected simple graph, $u, v \in V$, $S \subseteq V \backslash \{u, v\}$ and $F \subseteq E$.

- $S$ is a $(u, v)$-separating (vertex) set if $G \backslash S$ has no paths from $u$ to $v$.
- $F$ is a $(u, v)$-separating edge set if $G - F$ has no paths from $u$ to $v$.

Two paths from $u$ to $v$ are

- vertex-disjoint if their only common vertices are $u$ and $v$.
- edge-disjoint if they have no common edges.

## Theorem (Menger, 1929)

*Maximum number of pairwise edge-/vertex-disjoint paths from u to v is equal to the cardinality of minimum $(u, v)$-separating edge/vertex sets.*

The graph can be directed or undirected, thus we have 4 theorems here.

# Proof of Menger's theorem (edges, directed)

- Turn $G$ into network with source $u$ and sink $v$.
  - Delete edges going into $u$ or going out of $v$.
  - Give the capacity 1 to each edge.
- $(u, v)$-separating edge set of size $c \equiv$ cut of value $c$.
- Integral flow of value $c \equiv c$ edge-disjoint paths from $u$ to $v$.

# Proof of Menger's theorem (edge, undirected)

- Turn $G$ into network with source $u$ and sink $v$.
  - Edges incident to $u$ will be directed away from $u$.
  - Edges incident to $v$ will be directed towards $v$.
  - Other edges are replaced with directed edges in both directions.
  - The capacity of each edge is 1.
- $(u, v)$-separating edge set of size $c \equiv$ cut of value $c$.
- Integral flow of value $c \equiv c$ edge-disjoint paths from $u$ to $v$.

# Proof of Menger's theorems (vertices)

Do the same as for edges, but also

- Split each vertex $w$ (except $u$ and $v$) into two: $w_{\text{in}}$ and $w_{\text{out}}$, connected by an edge.
- Give capacity 1 to these edges. Give large capacities to all original edges of $G$.

# Chains and antichains in partially ordered sets

Let $(P, \leqslant)$ be a partially ordered set.

**Definition**

- $Q \subseteq P$ is a chain if $\forall x, y \in Q : (x \leqslant y \vee y \leqslant x)$.
- $Q \subseteq P$ is an antichain if $\forall x, y \in Q : x \neq y \Rightarrow (x \not\leqslant y \wedge y \not\leqslant x)$.

**Theorem (Dilworth, 1947)**

*If $m$ is the maximum cardinality of antichains in $P$, then $P$ can be partitioned into $m$ chains.*

# Proof of Dilworth's theorem

- Consider $|P| \times |P|$ matrix $M$, rows and columns indexed by $P$
  - The entry $(a, b)$ of $M$ equals 1 iff $a < b$
- A chain $a_1 < a_2 < \cdots < a_n$ gives us a partial transversal $\{(a_j, a_{j+1}) \,|\, j \in \{1, \ldots, n-1\}\}$ of size $n - 1$.
- A partition of $P$ to $k$ chains gives us a partial transversal of size $|P| - k$.
- Conversely, take the partition of $P$ to $|P|$ 1-element chains. Also take a partial transversal of size $|P| - k$.
  - Each "1" in it corresponds to a relation $a < b$ that can be used to join two chains.
  - Thus we get a partition of $P$ into $k$ chains.
- Let $m$ be minimal, such that $P$ can be partitioned into $m$ chains.
  - $|P| - m$ is the term rank of $M$
- There is a cover of $M$ by $|P| - m$ lines.
  - There are $m$ elements with corresponding row and column not in that cover.
  - These form an antichain in $P$.

# Doubly stochastic matrices

## Definition

A square matrix with entries from $\mathbb{R}_+$ is doubly stochastic if each row and each column of it sums up to 1.

## Definition

A square matrix with entries from $\{0, 1\}$ is a permutation matrix if each row and each column of it contains exactly one 1.

## Definition

A convex combination of objects $x_1, \ldots, x_k$ (supporting addition and multiplication with reals) is any object of the form $\lambda_1 x_1 + \cdots + \lambda_k x_k$, where $\lambda_i \geqslant 0$ and $\lambda_1 + \cdots + \lambda_k = 1$.

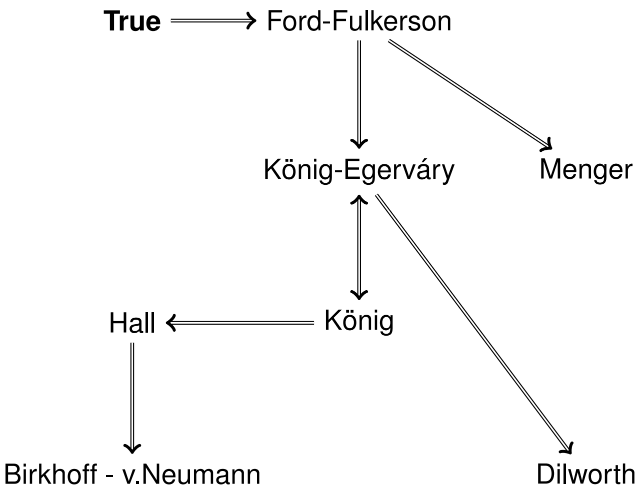
## Theorem (Birkhoff and von Neumann, 1946)

*Any doubly stochastic matrix can be expressed as a convex combination of permutation matrices.*

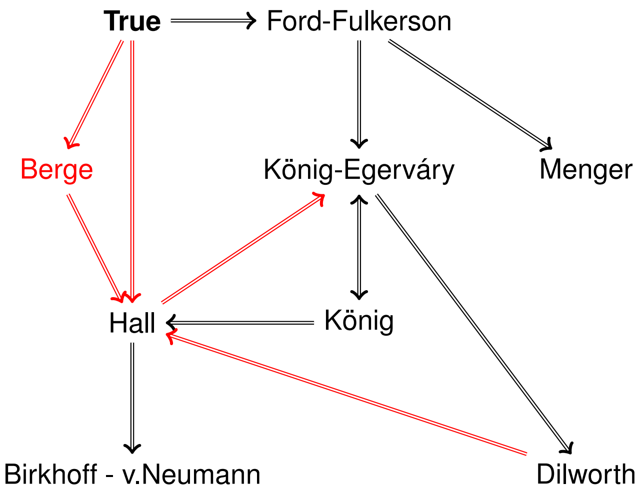
# Proof of Birkhoff's and von Neumann's theorem

- Let $M$ be a doubly stochastic matrix of size $n \times n$.
- Consider a bipartite graph $G = (X \dot{\cup} Y, E)$, where
  - $X = Y = \{1, \ldots, n\}$
  - There is an edge from $i \in X$ to $j \in Y$ iff $M_{ij} \neq 0$.
- $G$ satisfies Hall's criterion.
  - The entries in rows from any subset $S \subseteq X$ sum up to $|S|$. It takes at least $|S|$ columns to contain these entries.
- A matching covering all of $X$ gives us a permutation $\sigma$, such that $(i, \sigma(i))$ is a non-zero entry of $M$ for all $i$.
  - Let $\Sigma$ be the permutation matrix corresponding to $\sigma$.
- Let $\varepsilon$ be the minimum of these entries.
- $M = \varepsilon \cdot \Sigma + (1 - \varepsilon) \cdot M'$, where $M'$ is a doubly stochastic matrix with at least one more zero entry.
- We can do induction over the number of non-zero entries in $M$.

# The proofs we've done so far



# The proofs we'll still do



# A direct proof for Hall's theorem

- Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall's criterion.
- If $\forall x \in X : \deg(x) = 1$, then the matching is obvious.
- Let $x \in X$ be such that $\deg(x) \geqslant 2$. Let $(x, y_1), (x, y_2) \in E$.
- Assume that we can remove neither $(x, y_1)$ nor $(x, y_2)$ without violating Hall's criterion.
- There are $S_1, S_2 \subseteq X \backslash \{x\}$, such that

$$|N(S_i) \cup (N(x) \backslash \{y_i\})| < |S_i| + 1 \quad .$$

- Hence we get a contradiction:

$$\begin{aligned}
|S_1| + |S_2| &\geqslant |N(S_1) \cup (N(x) \backslash \{y_1\})| + |N(S_2) \cup (N(x) \backslash \{y_2\})| \geqslant \\
&\left| N(S_1) \cup (N(x) \backslash \{y_1\}) \cup N(S_2) \cup (N(x) \backslash \{y_2\}) \right| + \left| N(S_1) \cap N(S_2) \right| \geqslant \\
|N(S_1 \cup S_2 \cup \{x\})| + |N(S_1 \cap S_2)| &\geqslant |S_1 \cup S_2| + 1 + |S_1 \cap S_2| = |S_1| + |S_2| + 1
\end{aligned}$$

# Dilworth $\Rightarrow$ Hall

- Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall's criterion.
- Let $P = (X, cup Y, \leqslant)$ be a partially ordered set:
  - $x < y$ iff $x \in X$, $y \in Y$ and $(x, y) \in E$
- $Y$ is an antichain in $P$.
- If $Z$ is any antichain in $P$, then $N(Z \cap X) \cap (Z \cap Y) = \emptyset$. Hence

$$|Z| = |Z \cap X| + |Z \cap Y| \leqslant |N(Z \cap X)| + |Z \cap Y| \leqslant |Y| \ .$$

- $P$ can be partitioned to $|Y|$ chains.
- Each element of $X$ will be in a chain together with an element of $Y$. These give us the matching.

# Alternating paths

Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching in it.

## Definition

- An open path $P$ in $G$ is $M$-alternating ($M$-*vahelduv*) if the edges of $P$ alternatingly belong to $M$ and $E \setminus M$.
- An alternating path $P$ with endpoints $x$ and $y$ is $M$-augmenting ($M$-*laienev*) if $\deg_M(x) = \deg_M(y) = 0$.

## Theorem (Berge)

*A matching M in graph G is maximal iff there are no M-augmenting paths in G.*

# Proof of Berge's theorem

- ($\Rightarrow$): If $P$ is an $M$-augmenting path then $M' = (M \backslash P) \cup (P \backslash M)$ is a matching and $|M'| = |M| + 1$.
- ($\Leftarrow$): Let $M$ be a non-maximal matching in $G$. Let $M^*$ be a matching with $|M^*| > |M|$.
- Consider the graph $H = (V, M \cup M^*)$.
  - $\forall v \in V : \deg_H(v) \leqslant 2$.
- There are following kinds of connected components in $H$:
  - Isolated vertices.
  - Cycles (of even length).
  - Two vertices connected by an edge from $M \cap M^*$.
  - Paths, where edges from $M$ and $M^*$ alternate.
- There must be a connected component having more edges from $M^*$ than from $M$.
- Only possibility: path of odd length, starting and ending with an edge from $M^*$.
- This is an $M$-augmenting path.

# Berge $\Rightarrow$ Hall

- Let $G = (X \cup Y, E)$ be a bipartite graph. Let $M$ be a maximum matching in it. Let $x \in X$ be uncovered by $M$.
- Construct all possible $M$-alterating paths starting from $x$.
- Let $S \subset X$ be the set of vertices in $X$ on these paths (incl. $x$).
- Let $T \subset Y$ be the set of vertices in $Y$ on these paths.
- We have
  - $N(S) = T$, because any edge from some $u \in S$ can continue an $M$-alternating path.
  - $|S \setminus \{x\}| = |T|$. The edges in $M$ give a bijection between $S \setminus \{x\}$ and $T$.
    - The non-existence of $M$-augmenting paths implies that any $M$-alternating path ending in $Y$ can be continued.
- Hence $G$ does not satisfy Hall's criterion.

# Hall $\Rightarrow$ König-Egerváry

- Let $M$ be a $m \times n$ 0/1-matrix. Let its minimal cover consist of rows with indices in $R$ and columns with indices in $C$.
- Let $G_R = (R \cup \overline{C}, E_R)$, where $E_R = \{(r, c) \mid M_{rc} = 1\}$.
- Let $S \subseteq R$. Then $|N(S)| \geqslant S$, as otherwise the rows in $S$ could be replaced with the smaller number of columns in $N(S)$, still covering all 1-s in $M$.
- Thus $M$ has a partial transversal in rows $R$ and columns outside of $C$, such that its size is $|R|$.
- Similarly, let $G_C = (C \cup \overline{R}, E_C)$, where $E_C = \{(c, r) \mid M_{rc} = 1\}$.
- There's a partial transversal in columns $C$ and rows outside of $R$, such that its size is $|C|$.
- Joining these partial transversals, we get a partial transversal of size $|R| + |C|$.