Discrete Mathematics, 9th lecture

Binomial coefficients
Generating functions (1st part)

Peeter Laud

Cybernetica AS

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Two general rules on counting

$$|X_1 \times X_2 \times \cdots \times X_k| = |X_1| \cdot |X_2| \cdots |X_k|$$
$$|X_1 \,\dot{\cup}\, X_2 \,\dot{\cup}\, \cdots \,\dot{\cup}\, X_k| = |X_1| + |X_2| + \cdots + |X_k|$$

Permutations

Definition

Let X be a set, $|X| = n$. A sequence $(x_1, \ldots, x_k)$ is a permutation of X, if

- $x_1, \ldots, x_k \in X$;
- if $i \neq j$ then $x_i \neq x_j$.

Definition

Let $r \in \mathbb{C}$, $k \in \mathbb{N}$. The k-th falling factorial power of r (*Arvu r kahanev k-faktoriaal*) is $r^{\underline{k}} = r \cdot (r-1) \cdots (r-k+1)$.

I'd rather avoid the notation $(r)_k$ to reduce overloading.

Theorem

Let $P(n, m)$ be the number of permutations of size m of a set of cardinality n. Then $P(n, m) = n^{\underline{m}}$.

Combinations

Definition

$$\binom{n}{m} = |\{Y \subseteq \{1,\ldots,n\} \mid |Y| = m\}|$$

Here $m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $m \leqslant n$.

Theorem

$$\binom{n}{m} = \frac{n^{\underline{m}}}{m!} = \frac{n!}{m!(n-m)!}$$

Also, if $r, k \in \mathbb{N}$ then $r^{\underline{k}} = r!/(r-k)!$.

Pascal's triangle: values of $\binom{n}{m}$

Theorem

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} \qquad \text{if } n, m \in \mathbb{N}\backslash\{0\}, n \geqslant m$$

$n\backslash m$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Exercise: extend Pascal's triangle to arbitrary $n \in \mathbb{N}$, $m \in \mathbb{Z}$.

Generalizing $\binom{n}{m}$

$$\binom{n}{m} = \begin{cases} \frac{n^{\underline{m}}}{m!}, & \text{if } n \in \mathbb{N}, m \in \mathbb{N} \\ 0, & \text{if } n \in \mathbb{N}, m \in \mathbb{Z} \setminus \mathbb{N} \end{cases}$$

- $r^{\underline{m}}$ is defined for any $r \in \mathbb{C}$.
- So we could define

$$\binom{r}{m} = \begin{cases} \frac{r^{\underline{m}}}{m!}, & \text{if } n \in \mathbb{C}, m \in \mathbb{N} \\ 0, & \text{if } n \in \mathbb{C}, m \in \mathbb{Z} \setminus \mathbb{N} \end{cases}$$

Does the Pascal's triangle identity still hold? It does if $m \leqslant 0$. Otherwise...

Polynomial interpolation

Theorem

For any $(a_0, b_0), \ldots, (a_n, b_n) \in \mathbb{C}^2$ *with* a_i *mutually different, there is a unique polynomial* f *of degree at most* n*, such that*
$\forall i \in \{0, \ldots, n\} : f(a_i) = b_i$.

Proof.

Existence.

$$P_i(x) = \prod_{\substack{j=0\\j\neq i}}^{n} \frac{x - a_j}{a_i - a_j}$$

is a polynomial of degree n, such that $P_i(a_i) = 1$ and $P_i(a_j) = 0$, if $j \neq i$.
Take $f = \sum_{i=0}^{n} b_i P_i$.
Uniqueness. If f and f' are both such polynomials, then $(f - f')$ is a polynomial of degree $\leqslant n$, but with $\geqslant (n+1)$ roots. Hence it is constantly 0. □

Equality of polynomials

Corollary

If two polynomials of degree at most m agree on at least $m + 1$ different points, then they agree on the whole $\mathbb{C}$.

Equality of polynomials

Corollary

If two polynomials of degree at most m agree on at least $m+1$ different points, then they agree on the whole $\mathbb{C}$.

- $\binom{x}{m}$ is a polynomial of degree at most m.
- $\binom{x-1}{m} + \binom{x-1}{m-1}$ is a polynomial of degree at most m.
- They agree whenever $x \in \mathbb{N}\setminus\{0\}$.

Hence

$$\binom{r}{m} = \binom{r-1}{m} + \binom{r-1}{m-1} \text{ for any } r \in \mathbb{C}, m \in \mathbb{Z}$$

Extended Pascal's triangle

$n\backslash m$	-2	-1	0	1	2	3	4	5	6	7	8
-4	0	0	1	-4	10	-20	35	-56	84	-120	165
-3	0	0	1	-3	6	-10	15	-21	28	-36	45
-2	0	0	1	-2	3	-4	5	-6	7	-8	9
-1	0	0	1	-1	1	-1	1	-1	1	-1	1
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	0	0	0	0	0
2	0	0	1	2	1	0	0	0	0	0	0
3	0	0	1	3	3	1	0	0	0	0	0
4	0	0	1	4	6	4	1	0	0	0	0
5	0	0	1	5	10	10	5	1	0	0	0
6	0	0	1	6	15	20	15	6	1	0	0
7	0	0	1	7	21	35	35	21	7	1	0
8	0	0	1	8	28	56	70	56	28	8	1

Extended Pascal's triangle

Theorem

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k} \qquad r \in \mathbb{C}, k \in \mathbb{Z}$$

Lemma

$r^{\underline{k}} = (-1)^k (k-r-1)^{\underline{k}}$.

The theorem holds if $k < 0$. If $k \geqslant 0$, then use $\binom{r}{k} = r^{\underline{k}}/k!$ and the lemma.

Newton's binomial formula

Theorem

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m} \qquad x, y \in \mathbb{C}, n \in \mathbb{N}$$

Proof.

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdots(x+y)}_{n}$$

When opening the parentheses, there are $\binom{n}{m}$ ways to pick m times x and $(n-m)$ times y. □

Corollary

- $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$
- $\binom{n}{0} - \binom{n}{1} \pm \cdots + (-1)^n \binom{n}{n} = 0$, *if* $n \neq 0$

Generalization of Newton's binomial formula

Theorem

$$(x+y)^r = \sum_{m=0}^{\infty} \binom{r}{m} x^m y^{r-m} \qquad x, y, r \in \mathbb{C}, |x/y| < 1$$

- $|x/y| < 1$ is necessary for the absolute convergence of the sum.
- Cannot use polynomial argument to go from n to r because the sum is not a polynomial.

Theorem is equivalent to

$$(1+z)^r = \sum_{m=0}^{\infty} \binom{r}{m} z^m \qquad |z| < 1 \quad .$$

Absolute convergence of the sum is not too hard to show.

- If m is large then $\left|\binom{r}{m+1} z^{m+1}\right| \approx |z| \cdot \left|\binom{r}{m} z^m\right|$.

Generalization of Newton's binomial formula

Theorem

$$(x+y)^r = \sum_{m=-\infty}^{\infty} \binom{r}{m} x^m y^{r-m} \qquad x, y, r \in \mathbb{C}, |x/y| < 1$$

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Theorem is equivalent to

$$(1+z)^r = \sum_{m=0}^{\infty} \binom{r}{m} z^m \qquad |z| < 1 \quad .$$

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Proof of the generalized Newton's binomial formula

- Let $f(z) = (1+z)^r$.
- Note that $f'(z) = r(1+z)^{r-1}$, $f''(z) = r(r-1)(1+z)^{r-2}$
  - In general, $f^{(m)}(z) = r^{\underline{m}}(1-z)^{r-m}$.
- Expand f as Maclaurin series:

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m = \sum_{m=0}^{\infty} \frac{r^{\underline{m}}(1-0)^{r-m}}{m!} z^m = \sum_{m=0}^{\infty} \binom{r}{m} z^m$$

Many more simple identities

Identities can be proved using combinatorial or algebraic arguments.

- $\binom{r}{m} = \binom{r}{r-m}$
  - $\sum_{m=0}^{n} \binom{r+m}{m} = \binom{r+n+1}{n}$
  - $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$
- $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$
- $\binom{r+s}{m} = \sum_{k=0}^{m} \binom{r}{k}\binom{s}{m-k}$ (Vandermonde identity)
- $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$

Important special case of Newton's binomial formula

$$\begin{aligned}\frac{1}{(1-z)^{n+1}} &= \sum_{m=0}^{\infty}\binom{-n-1}{m}(-z)^m\\ &= \sum_{m=0}^{\infty}(-1)^m\binom{m+n+1-1}{m}(-1)^m z^m\\ &= \sum_{m=0}^{\infty}\binom{m+n}{m}z^m\end{aligned}$$

Multiplying both sides by z^n gives

$$\begin{aligned}\frac{z^n}{(1-z)^{n+1}} &= \sum_{m=-n}^{\infty}\binom{m+n}{m}z^{m+n}\\ &= \sum_{k=0}^{\infty}\binom{k}{k-n}z^k = \sum_{k=0}^{\infty}\binom{k}{n}z^k\end{aligned}$$

Permutations with repetitions

Definition

A multiset (or bag) is a pair (X, μ), where X is a set and $\mu : X \to \mathbb{N} \setminus \{0\}$.

- Think of $\mu(x)$ as the number of times x appears in X.
- The cardinality of (X, μ) is $\sum_{x \in X} \mu(x)$.

Definition

A permutation of a multiset (X, μ) is a sequence $(x_1, \ldots, x_m)$, where

- $\forall i : x_i \in X$;
- $\forall x \in X$: the number of occurrences of x in $(x_1, \ldots, x_m)$ is at most $\mu(x)$.

Theorem

Let $X = \{x_1, \ldots, x_k\}$, $\mu(x_i) = m_i$, $n = m_1 + \ldots + m_k$. The number of permutations of multiset (X, μ) of length n is $\binom{n}{m_1, m_2, \ldots, m_k} = \frac{n!}{m_1! \cdot m_2! \cdots m_k!}$.

Properties of multinomial coefficients

Theorem (Newton's multinomial formula)

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{m_1,\ldots,m_k \in \{0,\ldots,n\} \\ m_1+\cdots+m_k=n}} \binom{n}{m_1,\ldots,m_k} x_1^{m_1} \cdots x_k^{m_k}$$

Theorem

Let $n = m_1 + \cdots + m_k$

$$\binom{n}{m_1,\ldots,m_k} = \binom{n}{m_1} \cdot \binom{n-m_1}{m_2} \cdots \binom{n-m_1-\cdots-m_{k-2}}{m_{k-1}}$$

Combinations with repetitions

Let $|X| = n$. How many multisets (X', μ) of cardinality m exist, if $X' \subseteq X$? Denote by $F(n, m)$.

Theorem

$$F(n, m) = \binom{n + m - 1}{m}$$

Theorem

$$F(n, m) = F(n - 1, m) + F(n, m - 1)$$

Inversion

Theorem

Let $f, g : \mathbb{N} \to \mathbb{C}$.

$$g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) \Leftrightarrow f(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k g(k)$$

Permutations fixing a number of points

How many permutations π of n elements are there, such that $\pi(x) = x$ for exactly k different $x \in \{1, \ldots, n\}$?

- Let $h(n,k)$ be that number. We have
  - $n! = \sum_{k=0}^{n} h(n,k)$
  - $h(n,k) = \binom{n}{k} h(n-k, 0)$.
    - Denote $D_i = h(i, 0)$

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} D_{n-k} = \sum_{k=0}^{n} \binom{n}{k} D_k$$

$$D_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! = \sum_{k=0}^{n} (-1)^{n+k} \frac{n!}{(n-k)!} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Discrete Mathematics, 10th–11th lecture

Generating functions

Peeter Laud

Cybernetica AS

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Generating functions

Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence (with entries in $\mathbb{C}$). The (ordinary) generating function for $(a_n)_{n\in\mathbb{N}}$ is $A(z) = \sum_{n=0}^{\infty} a_n z^n$.

- The function (sum) A may be viewed in two ways
  - As an analytic function (converging if $|z|$ is sufficiently small)
  - as a formal sum, i.e. the sequence $a_0, a_1, a_2, \ldots$
- The first view introduces a lot of operations, simplifications, etc. on generating functions.
  - Addition, multiplication, composition, differentiation, integration, ...
  - Relations between them.
- They remain valid also for the second view.

Example 1

Evaluate $S = \sum_{i=0}^{k}(-1)^i\binom{i}{n}\binom{m}{k-i}$.

- Let $G(z) = \sum_{i=0}^{\infty}\binom{i}{n}z^i = \frac{z^n}{(1-z)^{n+1}}$.
- Let $H(z) = \sum_{i=0}^{\infty}\binom{m}{i}(-1)^{k+i}z^i = (-1)^k(1-z)^m$.
- The product of two sums is

$$\left(\sum_{i=0}^{\infty}\binom{i}{n}z^i\right)\cdot\left(\sum_{i=0}^{\infty}\binom{m}{i}(-1)^{m+i}z^i\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{i}{n}\binom{m}{k-i}(-1)^i\right)z^k$$

  when grouped by powers of z.
- $G(z)\cdot H(z) = z^n(-1)^k\cdot(1-z)^{m-n-1} = z^n\sum_{i=0}^{\infty}\binom{m-n-1}{i}(-1)^{i+k}z^i$
- Equating coefficients, we get $S = (-1)^n\binom{m-n-1}{k-n}$.

Example 2

Solve the recurrence $a_n = 2a_{n-1} - a_{n-2} + \binom{m+n}{m}$ with $a_0 = 1$, $a_1 = m$.

- The conditions as a single equation valid for all $n \in \mathbb{N}$:

$$a_n = 2a_{n-1} - a_{n-2} + \binom{m+n}{m} - 3 \cdot [n = 1]$$

  here $a_{-1} = a_{-2} = 0$, $[\textbf{true}] = 1$ and $[\textbf{false}] = 0$.
- Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$.
- Multiply both sides of recurrence with z^n. Add over all n.

$$\begin{aligned} A(z) &= \sum_{n=0}^{\infty} a_n z^n = 2\sum_{n=0}^{\infty} a_{n-1} z^n - \sum_{n=0}^{\infty} a_{n-2} z^n + \sum_{n=0}^{\infty} \binom{m+n}{m} z^n - 3z \\ &= 2z \sum_{n=0}^{\infty} a_n z^n - z^2 \sum_{n=0}^{\infty} a_n z^n + \frac{1}{(1-z)^{m+1}} - 3z \\ &= (2z - z^2) A(z) + \frac{1}{(1-z)^{m+1}} - 3z \end{aligned}$$

Example 2 - cont.

- Solve for $A(z)$

$$A(z) = \frac{1}{(1-z)^{m+3}} - \frac{3z}{(1-z)^2} = \sum_{n=0}^{\infty} \binom{n+m+2}{m+2} z^n - \sum_{n=0}^{\infty} 3(n+1) z^{n+1}$$

Hence $a_n = \binom{n+m+2}{m+2} - 3n = \binom{n+m+2}{n} - 3n$.

Example 3

Solve the recurrence $a_n = 1 + \sum_{i=0}^{n-1} a_i$, where $a_0 = 1$.

- The recurrence actually also determines a_0 (empty sum $\equiv$ 0).
- Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$.
- Multiply both sides of recurrence with z^n. Add over all n.

$$\begin{aligned}
A(z) &= \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty}\Big(1 + \sum_{i=0}^{n-1} a_i\Big) z^n = \Big(\sum_{n=0}^{\infty} z^n\Big) + \Big(\sum_{i=0}^{\infty} a_i \sum_{n=i+1}^{\infty} z^n\Big) \\
&= \frac{1}{1-z} + \Big(\sum_{i=0}^{\infty} a_i z^i\Big) \cdot \Big(\sum_{n=1}^{\infty} z^n\Big) \\
&= \frac{1}{1-z} + A(z)\frac{z}{1-z}
\end{aligned}$$

- Hence $A(z) = \frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n$.
- $a_n = 2^n$

Example 4

How many ways are there to tile a $2 \times n$ rectangle with 2×1 dominoes, such that there are m “horizontal” dominoes?

- Let $h_{n,m}$ be the correct number.
  - $h_{n,0} = 1$. $h_{0,m} = 0$ if $m > 0$.
  - $h_{2n,2n} = 1$. $h_{2n+1,2n} = n + 1$.
  - $h_{n,2m+1} = 0$. $h_{n,m} = 0$ if $m > n$.
- $h_{n,m} = h_{n-1,m} + h_{n-2,m-2}$ if $n \geqslant 2$, $m \geqslant 2$.
- Try to write equation for $h_{n,m}$ that combines those and is valid for all $m, n \in \mathbb{N}$, assuming $h_{n,m} = 0$ if $n < 0$ or $m < 0$.

$$h_{n,m} = h_{n-1,m} + h_{n-2,m-2} + [n = 0 \wedge m = 0]$$

- multiply both sides with $z^n w^m$ and sum over all n and m.

Example 4 (cont.)

$$
\begin{aligned}
H(z,w) &= \sum_{m,n=0}^{\infty} h_{n,m} z^n w^m = \sum_{m,n=0}^{\infty} h_{n-1,m} z^n w^m + \sum_{m,n=0}^{\infty} h_{n-2,m-2} z^n w^m + 1 \\
&= zH(z,w) + z^2 w^2 H(z,w) + 1 \\
H(z,w) &= \frac{1}{1 - z - z^2 w^2} = \sum_{k=0}^{\infty} (z + z^2 w^2)^k = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} z^{2k-i} w^{2(k-i)} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [m \text{ is even}] \binom{n - m/2}{n - m} z^n w^m
\end{aligned}
$$

Hence the number of tilings is $\binom{n-m/2}{n-m} = \binom{n-m/2}{m/2}$ (only if m is even).

Example 5

How many ways are there to tile a $3 \times n$ rectangle with 2×1 dominoes, such that there are m "horizontal" dominoes?

- Let $u_{n,m}$ be the correct number.
- In general: $u_{n,m} = u_{n-2,m-3} + 2v_{n,m}$.
- $v_{n,m}$ — number of ways to tile a $3 \times (n-1)$ rectangle plus an extra corner with dominoes, including m horizontal.
- $v_{n,m} = u_{n-2,m-1} + v_{n-2,m-3}$.
- Corner cases:
    - $u_{2n+1,m} = v_{2n+1,m} = 0$
    - $u_{0,0} = 1$. $u_{0,m} = 0$ if $m > 0$. $v_{0,m} = 0$.

$$\begin{aligned} u_{n,m} &= u_{n-2,m-3} + 2v_{n,m} + [n = 0 \wedge m = 0] \\ v_{n,m} &= u_{n-2,m-1} + v_{n-2,m-3} \end{aligned}$$

Let $U(z,w) = \sum_{m,n=0}^{\infty} u_{n,m} z^n w^m$ and $V(z,w) = \sum_{m,n=0}^{\infty} v_{n,m} z^n w^m$.

Example 5 (cont.)

$$\begin{aligned} U(z,w) &= \sum_{m,n=0}^{\infty} u_{n-2,m-3} z^n w^m + 2 \sum_{m,n=0}^{\infty} v_{n,m} z^n w^m + 1 \\ &= z^2 w^3 U(z,w) + 2V(z,w) + 1 \\ V(z,w) &= \sum_{m,n=0}^{\infty} u_{n-2,m-1} z^n w^m + \sum_{m,n=0}^{\infty} v_{n-2,m-3} z^n w^m \\ &= z^2 w U(z,w) + z^2 w^3 V(z,w) \end{aligned}$$

Solving for U gives

$$U(z,w) = \frac{1 - z^2 w^3}{(1 - z^2 w^3)^2 - 2z^2 w}$$

this can be manipulated as. . .

Example 5 (cont.)

$$
\begin{aligned}
U(z,w) &= \frac{1-z^2w^3}{(1-z^2w^3)^2-2z^2w} = \frac{(1-z^2w^3)^{-1}}{1-2z^2w(1-z^2w^3)^{-2}} \\
&= \frac{1}{1-z^2w^3}\sum_{k=0}^{\infty}\left(\frac{2z^2w}{(1-z^2w^3)^2}\right)^k = \sum_{k=0}^{\infty}\frac{2^k z^{2k} w^k}{(1-z^2w^3)^{2k+1}} \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{2k+i}{2k}2^k z^{2k+2i} w^{k+3i} \\
&= \sum_{n,m=0}^{\infty}[n \text{ and } m-n/2 \text{ are even}]\binom{(5n-2m)/4}{(3n-2m)/2}2^{(3n-2m)/4} z^n w^m
\end{aligned}
$$

Example 6

A permutation π is an involution if $\pi = \pi^{-1}$. How many involutions of n elements are there?

- Let a_n be the correct number. Then $a_n = a_{n-1} + (n-1)a_{n-2} + [n = 0]$.
- Multiply both sides with z^n, sum over n.

$$
\begin{aligned}
A(z) &= \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_{n-1} z^n + \sum_{n=0}^{\infty} (n-1) a_{n-2} z^n + 1 \\
&= zA(z) + z^2 + z^3 \sum_{n=1}^{\infty} (n+1) a_n z^{n-1} + 1 \\
&= zA(z) = z^2 + z^3 A'(z) + z^2 (A(z) - 1) + 1 \\
&= (z + z^2) A(z) + z^3 A'(z) + 1 \\
-\frac{1}{z^3} &= A'(z) + \frac{z^2 + z - 1}{z^3} A(z)
\end{aligned}
$$

Linear 1st order inhomogeneous diff. eq.-s

http://en.wikipedia.org/wiki/Linear_differential_equation

General form:

$$y' + f(x)y = g(x)$$

General solution:

$$y = \mathrm{e}^{-a(x)}\left(\int g(x)\mathrm{e}^{a(x)}\,dx + C\right) \text{ where } a(x) = \int f(x)\,dx$$

It's probably hard to read out the coefficients of z^n from this expression

Example 6'

Let $b_n = a_n/n!$. Then $b_n = (b_{n-1} + b_{n-2})/n$ if $n > 0$.

$$\begin{aligned} \sum_{n=1}^{\infty} n b_n z^n &= \sum_{n=1}^{\infty} b_{n-1} z^n + \sum_{n=1}^{\infty} b_{n-2} z^n \\ zB'(z) &= zB(z) + z^2 B(z) \end{aligned}$$

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\end{aligned}$$

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Let $b_n = a_n/n!$. Then $b_n = (b_{n-1} + b_{n-2})/n$ if $n > 0$.

$$\begin{aligned}
\sum_{n=1}^{\infty} n b_n z^n &= \sum_{n=1}^{\infty} b_{n-1} z^n + \sum_{n=1}^{\infty} b_{n-2} z^n \\
B'(z) &= B(z) + z\ B(z) \\
\frac{dB}{dz} &= (1+z)B \\
\frac{dB}{B} &= (1+z)\,dz
\end{aligned}$$

Example 6'

Let $b_n = a_n/n!$. Then $b_n = (b_{n-1} + b_{n-2})/n$ if $n > 0$.

$$\begin{aligned}
\sum_{n=1}^{\infty} n b_n z^n &= \sum_{n=1}^{\infty} b_{n-1} z^n + \sum_{n=1}^{\infty} b_{n-2} z^n \\
B'(z) &= B(z) + z\ B(z) \\
\frac{dB}{dz} &= (1+z)B \\
\int \frac{dB}{B} &= \int (1+z)\, dz
\end{aligned}$$

Example 6'

Let $b_n = a_n/n!$. Then $b_n = (b_{n-1} + b_{n-2})/n$ if $n > 0$.

$$\begin{aligned}
\sum_{n=1}^{\infty} n b_n z^n &= \sum_{n=1}^{\infty} b_{n-1} z^n + \sum_{n=1}^{\infty} b_{n-2} z^n \\
B'(z) &= B(z) + z\ B(z) \\
\frac{dB}{dz} &= (1+z)B \\
\int \frac{dB}{B} &= \int (1+z)\,dz \\
\ln B &= z + z^2/2 + C \\
B(z) &= \mathrm{e}^{z+z^2/2}
\end{aligned}$$

Here $C = 0$ because $\ln B(0) = \ln b_0 = \ln 1 = 0$.

Example 6' (cont.)

$\mathrm{e}^z = \sum_{i=0}^{\infty} z^i / i!$.

$$
\begin{aligned}
\mathrm{e}^{z+z^2/2} &= \sum_{n=0}^{\infty} \frac{1}{n!}(z + z^/2)^n = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{1}{2^k n!} \binom{n}{k} z^{n+k} \\
&= \sum_{m=0}^{\infty} \sum_{n=\lceil m/2 \rceil}^{\infty} \frac{1}{2^{m-n} n!} \binom{n}{m-n} z^m \\
[z^n]B(z) &= \sum_{i=\lceil n/2 \rceil}^{n} \frac{1}{2^{n-i} i!} \binom{i}{n-i} \\
a_n = n![z^n]B(z) &= \sum_{i=\lceil n/2 \rceil}^{n} \frac{n^{\underline{i}}}{2^{n-i}(2i-n)!} = \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} \frac{i^{\underline{n-i}}}{2^{n-i}}
\end{aligned}
$$

The last sum could be summed from $-\infty$ to ∞.

Example 7

How many legal strings of n pairs of parentheses are there?

- Let c_n be the number. $c_0 = 1$.
- By considering the shortest non-empty prefixes of legal strings that are themselves legal, we get $c_n = \sum_{k=1}^{n} c_{k-1} c_{n-k} + [n = 0]$.

$$\begin{aligned} C(z) &= \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} \Bigl(\sum_{k=1}^{n} c_{k-1} c_{n-k} \Bigr) z^n + 1 = z(C(z))^2 + 1 \\ C(z) &= \frac{1 \pm \sqrt{1 - 4z}}{2z} \end{aligned}$$

At the point $z = 0$, the numerator should be 0, because $C(0) = c_0$ is finite. Hence $C(z) = (1 - \sqrt{1 - 4z})/2z$.

Example 7 (cont.)

$$\begin{aligned}
\sqrt{1-4z} &= \sum_{n=0}^{\infty}\binom{1/2}{n}(-4)^n z^n = 1 - 4z\sum_{n=1}^{\infty}\frac{1}{2n}\binom{-1/2}{n-1}(-4)^{n-1}z^{n-1} \\
C(z) &= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{-1/2}{n}(-4)^n z^n = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{z^n}{n+1}
\end{aligned}$$

because

$$\begin{aligned}
\binom{-1/2}{n} &= \frac{(-1/2)\cdot(-3/2)\cdots(-(2n-1)/2)}{n!} = \left(-\frac{1}{2}\right)^n \frac{1\cdot 3\cdots(2n-1)}{n!} \\
&= \left(-\frac{1}{4}\right)^n \frac{1\cdot 3\cdots(2n-1)}{n!}\cdot\frac{2\cdot 4\cdots(2n)}{n!} = \left(-\frac{1}{4}\right)^n \frac{(2n)!}{n!\cdot n!} \\
&= \left(-\frac{1}{4}\right)^n \binom{2n}{n}
\end{aligned}$$

Example 8

- Fibonacci numbers: $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$.
- Generating function: $F(z) = z/(1 - z - z^2)$.

Show that $f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$.

- GF of the LHS: $F(z)/(1 - z)$.
- GF of the RHS: $(F(z) - z)/z^2 - 1/(1 - z)$.

These GFs are equal.

Formal power series

Definition

A formal power series is a sequence $A = (a_n)_{n\in\mathbb{N}}$, of complex numbers. A is polynomial if it has finite number of non-zero entries.

- Think of A as a sum $A(z) = \sum_{n=0}^{\infty} a_n z^n$.
- Denote a_n also by $[z^n]A(z)$.

Definition

Operations with formal power series:

$$[z^n](A+B)(z) = [z^n]A(z) + [z^n]B(z)$$

$$[z^n](kA)(z) = k \cdot [z^n]A(z)$$

$$[z^n](A \cdot B)(z) = \sum_{i=0}^{n} [z^i]A(z) \cdot [z^{n-i}]B(z)$$

$$A^k = \underbrace{A \cdot A \cdots A}_{k}$$

More operations

Reciprocal

A^{-1} is a FPS, such that $A \cdot A^{-1} = A^{-1} \cdot A = 1$. It exists iff $[z^0]A(z) \neq 0$.

Composition

$$A \circ B = \sum_{n=0}^{\infty} [z^n]A(z) \cdot B^n$$

It is well-defined if A is polynomial, or $[z^0]B(z) = 0$.
Exercise. Define A^r, where $r \in \mathbb{C}$.

Differentiation and integration

$$[z^n]A'(z) = (n+1) \cdot [z^{n+1}]A(z)$$
$$[z^n] \int A(z) = \frac{1}{n}[z^{n-1}]A(z)$$

Sum of a subseries

- Find $\sum_{n=0}^{\infty} r^n \binom{m}{n}$.
  - Answer: $(1+r)^m$.
- Find $\sum_{n=0}^{\infty} r^{2n} \binom{m}{2n}$.
  - I.e. we want to take every second component of the previous sum.

$$\text{If } A(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ then } A(z) + A(-z) = 2 \sum_{n=0}^{\infty} a_{2n} z^{2n} \quad .$$

Answer: $\big((1+r)^m + (1-r)^m\big)/2$.

- Find $\sum_{n=0}^{\infty} r^{3n} \binom{m}{3n}$.

Taking every third element

$$\frac{1^n + (-1)^n}{2} = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Taking every third element

$$\frac{1^n + (-1)^n}{2} = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$\frac{1^n + \omega^n + \omega^{2n}}{3} = \begin{cases} 1, & \text{if } n \text{ is divisible by } 3 \\ 0, & \text{if } n \text{ is not divisible by } 3 \end{cases}$$

where $\omega = (-1 + \sqrt{3}\mathrm{i})/2$. Then $\omega^2 = (-1 - \sqrt{3}\mathrm{i})/2$.
Hence

$$3 \sum_{n=0}^{\infty} a_{3n} z^{3n} = A(z) + A(\omega z) + A(\omega^2 z)$$

$$\sum_{n=0}^{\infty} r^{3n} \binom{m}{3n} = \frac{(1+r)^m + (1+\omega r)^m + (1+\omega^2 r)^m}{3}$$

An interesting identity

Taking $r = -1$ in the previous identity gives us

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\binom{m}{3n} &= \sum_{n=0}^{\infty}(-1)^{3n}\binom{m}{3n} = \left((1-\omega)^m + (1-\omega^2)^m\right)/3 \\
&= \frac{1}{3}\left(\left(\frac{3-\sqrt{3}\mathrm{i}}{2}\right)^m + \left(\frac{3+\sqrt{3}\mathrm{i}}{2}\right)^m\right) \\
&= \frac{(\sqrt{3})^m}{3}\left((\cos 30^\circ - \mathrm{i}\sin 30^\circ)^m + (\cos 30^\circ + \mathrm{i}\sin 30^\circ)^m\right) \\
&= 2 \cdot 3^{m/2-1}\cos(m \cdot 30^\circ)
\end{aligned}
$$

Inversion again

Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be sequences. Then

$$g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k \Leftrightarrow f_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k g_k$$

Let F, G be corresponding generating functions. Then

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} g_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k z^n \\
&= \sum_{k=0}^{\infty} (-1)^k f_k \sum_{n=k}^{\infty} \binom{n}{k} z^n = \sum_{k=0}^{\infty} (-1)^k f_k z^k \frac{1}{(1-z)^{k+1}} \\
&= \frac{1}{1-z} F\left(\frac{-z}{1-z}\right)
\end{aligned}$$

Inversion states: $G(z) = F(-z/(1-z))/(1-z)$ iff $F(z) = G(-z/(1-z))/(1-z)$.

Inversion with exponential generating functions

Let $F(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$ and $G(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$ be exponential generating functions of $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$. Then

$$
\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} \frac{g_n z^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k f_k z^n}{n!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)^k f_k z^n}{k!(n-k)!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^k f_k z^{n+k}}{k!n!} \\
&= \left(\sum_{k=0}^{\infty} f_k \frac{(-z)^k}{k!} \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) = \mathrm{e}^z F(-z)
\end{aligned}
$$

Inversion states: $G(z) = \mathrm{e}^z F(-z)$ iff $F(z) = \mathrm{e}^z G(-z)$.

Exponential generating functions

- Formally: sequences of integers $(a_n)_{n\in\mathbb{N}}$.
- Interpreted as formal sums $A(z) = \sum_{n=0}^{\infty} a_n z^n/n!$.
- Denote $[z^n/n!]A(z) = a_n$.

Exercise. What are

- $[z^n/n!](A+B)(z)$,
- $[z^n/n!](A\cdot B)(z)$,
- $[z^n/n!]A'(z)$?

Answers to the exercise

$$\begin{aligned}
\left[\frac{z^n}{n!}\right](A+B)(z) &= \left[\frac{z^n}{n!}\right]A(z) + \left[\frac{z^n}{n!}\right]B(z) \\
\left[\frac{z^n}{n!}\right](A\cdot B)(z) &= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{z^k}{k!}\right]A(z)\cdot\left[\frac{z^{n-k}}{(n-k)!}\right]B(z) \\
\left[\frac{z^n}{n!}\right]A'(z) &= \left[\frac{z^{n+1}}{(n+1)!}\right]A(z)
\end{aligned}$$

Example: Exponential generating function for Fibonacci numbers

$f_n = f_{n-1} + f_{n-2} + [n = 1]$. Let $F(z) = \sum_{n=0}^{\infty} f_n z^n / n!$.

$$F(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} f_{n-1} \frac{z^n}{n!} + \sum_{n=0}^{\infty} f_{n-2} \frac{z^n}{n!} + x$$

$$F''(z) = \sum_{n=0}^{\infty} f_{n+1} \frac{z^n}{n!} + \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} = F'(z) + F(z)$$

or $F'' - F' - F = 0$.

Linear homogeneous diff. eq.s with constant coeff.s

http://en.wikipedia.org/wiki/Linear_differential_equation

Let the solutions of

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

be $c_1, \ldots c_k$ with multiplicities $r_1, \ldots, r_k$ (then $r_1 + \cdots + r_k = n$).

Then the solutions of

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

are linear combinations of functions of the form

$$x^t \mathrm{e}^{c_j x} \qquad \left(j \in \{1, \ldots, k\}, t \in \{0, \ldots, r_j - 1\}\right) \ .$$

EGF for Fibonacci numbers (cont.)

$$x^2 - x - 1 = (x - \phi)(x - \overline{\phi}), \qquad \phi, \overline{\phi} = \frac{1 \pm \sqrt{5}}{2}$$

$$F(z) = C_1 \mathrm{e}^{\phi z} + C_2 \mathrm{e}^{\overline{\phi} z}$$

Initial conditions $0 = f_0 = F(0)$ and $1 = f_1 = F'(0)$ give us $C_1 = 1/\sqrt{5}$ and $C_2 = -1/\sqrt{5}$.

$$\begin{aligned} F(z) &= \frac{\mathrm{e}^{\phi z} - \mathrm{e}^{\overline{\phi} z}}{\sqrt{5}} \\ \left[\frac{z^n}{n!}\right] F(z) &= \frac{1}{\sqrt{5}} (\phi^n - \overline{\phi}^n) \end{aligned}$$

Proving an identity of Fibonacci numbers

$$\sum_{i=0}^{n} f_i \binom{n}{i} = f_{2n}$$

(*Reimo Palm, Diskreetse Matemaatika Elemendid, Ex. IV-10*)

- LHS equals $[z^n/n!]F(z) \cdot e^z$.
  - e^z is the EGF of $(1, 1, 1, \ldots)$

$$\begin{aligned} F(z)e^z &= \frac{\mathrm{e}^{(\phi+1)z} - \mathrm{e}^{(\overline{\phi}+1)z}}{\sqrt{5}} = \frac{\mathrm{e}^{\phi^2 z} - \mathrm{e}^{\overline{\phi}^2 z}}{\sqrt{5}} \\ \left[\frac{z^n}{n!}\right] F(z)e^z &= \frac{1}{\sqrt{5}}(\phi^{2n} - \overline{\phi}^{2n}) = f_{2n} \end{aligned}$$

Objects with a number of properties

Let X be a set. Let there be r predicates $\mathbb{P}_1, \ldots, \mathbb{P}_r$ given on X. Let

- p_C, where $C \subseteq \{1, \ldots, r\}$ be the number of elements satisfying all $\mathbb{P}_i$ for $i \in C$;
- $p_n = \sum_{|C|=n} p_C$;
- q_C, where $C \subseteq \{1, \ldots, r\}$ be the number of elements satisfying all $\mathbb{P}_i$ for $i \in C$ and none of $\mathbb{P}_j$ for $j \notin C$;
- $q_n = \sum_{|C|=n} q_C$ be the number of elements satisfying exactly n predicates.

$$\begin{aligned}
p_C &= \sum_{C' \supseteq C} q_{C'} \\
p_n &= \sum_{|C|=n} \sum_{C' \supseteq C} q_{C'} = \sum_{k=n}^{r} \binom{k}{n} \sum_{|C'|=k} q_{C'} = \sum_{k=n}^{r} \binom{k}{n} q_k
\end{aligned}$$

Let P, Q be ordinary GF-s of $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$.

Principle of inclusion-exclusion (p.i.e)

elimineerimismeetod

$$\begin{aligned} P(z) &= \sum_{n=0}^{r} p_n z^n = \sum_{n=0}^{r}\sum_{k=n}^{r}\binom{k}{n} q_k z^n = \sum_{k=0}^{r} q_k \sum_{n=0}^{k}\binom{k}{n} z^n = \sum_{n=0}^{k} q_k (1+z)^k \\ &= Q(z+1) \end{aligned}$$

The number of objects with no properties is

$$q_0 = Q(0) = P(-1) = \sum_{n=0}^{r} (-1)^n p_n$$

The number of objects with exactly m properties is

$$q_m = \frac{Q^{(m)}(0)}{m!} = \frac{P^{(m)}(-1)}{m!} = \sum_{n=m}^{r} (-1)^{n-m} \frac{n^{\underline{m}}}{m!} p_n = \sum_{n=m}^{r} (-1)^{n-m} \binom{n}{m} p_n$$

Number of surjective functions

How many surjective functions are there from $\{1,\ldots,s\}$ to $\{1,\ldots,t\}$?

- X — all functions from $\{1,\ldots,s\}$ to $\{1,\ldots,t\}$?
- Define $\mathbb{P}_i$ by $\mathbb{P}_i(f) \Leftrightarrow \nexists j : f(j) = i$. ($i \in \{1,\ldots,t\}$)
- $p_C = (t - |C|)^s$.
- $p_n = \binom{t}{n}(t-n)^s$.
- The number of surjective functions is

$$q_0 = \sum_{n=0}^{t} (-1)^n \binom{t}{n} (t-n)^s$$

The number of partitions of s-element set into t parts is

$$\left\{ {s \atop t} \right\} = \frac{q_0}{t!} = \sum_{n=0}^{t} \frac{(-1)^n (t-n)^s}{n!(t-n)!} = \sum_{n=0}^{t} \frac{(-1)^{t-n} n^s}{n!(t-n)!}$$

(Stirling numbers of second kind)

Stirling numbers of second kind

“Mixed” generating function:

$$\begin{aligned}
S(z,w) &= \sum_{s,t=0}^{\infty} \left\{ {s \atop t} \right\} \frac{z^s}{s!} w^t = \sum_{s,t=0} \sum_{n=0}^{t} \frac{(-1)^{t-n} n^s}{n!(t-n)!} \frac{t!}{t!} \frac{z^s}{s!} w^t \\
&= \sum_{t=0}^{\infty} \frac{(-w)^t}{t!} \sum_{n=0}^{t} (-1)^n \binom{t}{n} \sum_{s=0}^{\infty} \frac{(nz)^s}{s!} \\
&= \sum_{t=0}^{\infty} \frac{(-w)^t}{t!} \sum_{n=0}^{t} (-1)^n \binom{t}{n} \mathrm{e}^{nz} = \sum_{t=0}^{\infty} \frac{(-w)^t}{t!} (1-\mathrm{e}^z)^t \\
&= \mathrm{e}^{w(\mathrm{e}^z - 1)}
\end{aligned}$$

We can find a recurrent relation from it:

Stirling numbers of second kind

$$e^{w(e^z-1)} = \sum_{s,t=0}^{\infty} \left\{ {s \atop t} \right\} \frac{z^s}{s!} w^t$$

Take logarithm of both sides

$$w(e^z - 1) = \ln \left(\sum_{s,t=0}^{\infty} \left\{ {s \atop t} \right\} \frac{z^s}{s!} w^t \right)$$

Differentiate with respect to z

$$we^z = \frac{\sum_{s=1}^{\infty} \sum_{t=0}^{\infty} \left\{ {s \atop t} \right\} \frac{z^{s-1}}{(s-1)!} w^t}{\sum_{s,t=0}^{\infty} \left\{ {s \atop t} \right\} \frac{z^s}{s!} w^t} = \frac{\sum_{s,t=0}^{\infty} \left\{ {s+1 \atop t} \right\} \frac{z^s}{s!} w^t}{\sum_{s,t=0}^{\infty} \left\{ {s \atop t} \right\} \frac{z^s}{s!} w^t}$$

Clear the fractions

Stirling numbers of second kind

$$\begin{aligned}
\sum_{s,t=0}^{\infty} \left\{ {s \atop t} \right\} \frac{z^s}{s!} w^{t+1} e^z &= \sum_{s,t=0}^{\infty} \left\{ {s+1 \atop t} \right\} \frac{z^s}{s!} w^t \\
\left[\frac{z^s}{s!} w^{t+1} \right] LHS &= \sum_{n=0}^{s} \binom{s}{n} \left\{ {n \atop t} \right\} \\
\left[\frac{z^s}{s!} w^{t+1} \right] RHS &= \left\{ {s+1 \atop t+1} \right\}
\end{aligned}$$

- We have found $\left\{ {s+1 \atop t+1} \right\} = \sum_{n=0}^{s} \binom{s}{n} \left\{ {n \atop t} \right\}$.
- ...which has a nice combinatorial interpretation.

Exercise. Show that $\left\{ {s \atop t} \right\} = t \left\{ {s-1 \atop t} \right\} + \left\{ {s-1 \atop t-1} \right\}$.

Connected labeled graphs

How many connected labeled graphs with n vertices are there?

- Let d_n be the answer. Let h_n be the number of labeled graphs with n vertices.
  - $h_n = 2^{\binom{n}{2}}$.
- Let $D(z)$ and $H(z)$ be the respective EGF-s.
- By considering the connected component of the vertex labeled n, we get

$$\begin{aligned} h_n &= \sum_{k=0}^{n} \binom{n-1}{k-1} d_k h_{n-k} \\ n h_n &= \sum_{k=0}^{n} \binom{n}{k} k d_k h_{n-k} \qquad \left(\text{because } \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}\right) \end{aligned}$$

This gives us recursive formula for d_n. Let us also find the generating function.

EGFs of nh_n and d_n

http://en.wikipedia.org/wiki/Exponential_formula

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n \frac{z^n}{n!} &= H(z) \\
\sum_{n=0}^{\infty} nh_n \frac{z^n}{n!} &= z \sum_{n=1}^{\infty} nh_n \frac{z^{n-1}}{n!} = z \left(\sum_{n=1}^{\infty} h_n \frac{z^n}{n!} \right)' = zH'(z)
\end{aligned}$$

$$\begin{aligned}
zH'(z) &= zD'(z)H(z) \\
D'(z) &= \frac{H'(z)}{H(z)} \\
D(z) &= \ln H(z) + C
\end{aligned}$$

and $C = 1$ because $d_0 = D(0) = 1$ and $h_0 = H(0) = 1$.

Making change

There are coins of size $c_1, \ldots, c_k$ cents. How many ways are there to pay n cents?

- Let $C_i(z_i) = 1 + z_i^{c_i} + z_i^{2c_i} + \cdots = 1/(1 - z_i^{c_i})$.
- $[z^m]C_i(z)$ is the number of ways m cents can be payed with c_i-cent coins only.
- $[z_1^{m_1 c_1} \cdots z_k^{m_k c_k}]C_1(z_1) \cdots C_k(z_k)$ is the number of ways $m_1 c_1 + \cdots + m_k c_k$ cents can be payed using m_1 c_1-cent coins, m_2 c_2-cent coins, etc.
- If we set $z_1 = \cdots = z_m = z$, then $[z^n]C_1(z) \cdots C_k(z)$ counts the number of ways n cents can be payed in any manner with coins of worth $c_1, \ldots, c_k$ cents.
- The ordinary generating function is

$$C(z) = \frac{1}{(1 - z^{c_1})(1 - z^{c_2}) \cdots (1 - z^{c_k})} = \sum_{n=0}^{\infty} d_n z^n$$

A recurrent formula

Let us differentiate…

$(1/(1-z^c))' = [-1/(1-z^c)^2] \cdot [-cz^{c-1}] = \frac{cz^{c-1}}{1-z^c} \cdot \frac{1}{1-z^c}$

$C'(z) = \left(\sum_{i=1}^{k} \frac{c_i z^{c_i-1}}{1-z^{c_i}}\right) C(z)$

$$\sum_{n=0}^{\infty}(n+1)d_{n+1}z^n = \left(\sum_{i=1}^{k}\sum_{n=0}^{\infty} c_i[n \bmod c_i = -1]z^n\right)\left(\sum_{n=0}^{\infty} d_n z^n\right)$$

Let $a_n = |\{(i,j) \,|\, i \in \{1,\ldots,k\}, j \in \{1,\ldots,c_i\}, n \bmod c_i = -1\}|$.
Note that $a_n = a_{n-u}$, where $u = \mathrm{lcm}(c_1,\ldots,c_k)$.

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)d_{n+1}z^n &= \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} d_n z^n\right) \\ d_n &= \frac{1}{n}\sum_{l=0}^{n-1} a_l d_{n-l-1} = \frac{1}{n}\left(\sum_{l=0}^{u-1} a_l d_{n-l-1} + (n-u)d_{n-u}\right)\end{aligned}$$

OGFs of certain sequences

$a_0, a_1, a_2, a_3, \ldots$	a_n	$\sum_{n=0}^{\infty} a_n z^n$
$1, 0, 0, 0, \ldots$	$[n = 0]$	1
$\underbrace{0, \ldots, 0}_{m}, 1, 0, 0, \ldots$	$[n = m]$	z^m
$1, 1, 1, 1, \ldots$	1	$1/(1-z)$
$1, c, c^2, c^3, \ldots$	c^n	$1/(1-cz)$
$1, \underbrace{0, \ldots, 0}_{m-1}, 1, \underbrace{0, \ldots, 0}_{m-1}, 1, \ldots$	$[m \mid n]$	$1/(1-z^m)$
$1, 2, 3, 4, \ldots$	n	$1/(1-z)^2$
$\binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \ldots$	$\binom{r}{n}$	$(1+z)^r$
$\binom{r}{r}, \binom{r+1}{r}, \binom{r+2}{r}, \binom{r+3}{r}, \ldots$	$\binom{r+n}{r} = \binom{r+n}{n}$	$1/(1-z)^{r+1}$
$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$	$[n \neq 0]/n$	$\ln \frac{1}{1-z}$
$0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$	$(-1)^{n+1}[n \neq 0]/n$	$\ln(1+z)$
$1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots$	$\frac{1}{n!}$	e^z

OGFs of certain sequences

$a_0, a_1, a_2, a_3, \ldots$	a_n	$\sum_{n=0}^{\infty} a_n z^n$
$0, 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \ldots$	H_n	$\frac{1}{1-z} \ln \frac{1}{1-z}$
$F_0, F_m, F_{2m}, F_{3m}, \ldots$	F_{nm}	$\frac{F_m z}{1-(F_{m-1}+F_{m+1})z+(-1)^m z^2}$
$\left\{ {0 \atop m} \right\}, \left\{ {1 \atop m} \right\}, \left\{ {2 \atop m} \right\}, \left\{ {3 \atop m} \right\}, \ldots$	$\left\{ {n \atop m} \right\}$	$\frac{z^m}{(1-z)(1-2z)\cdots(1-mz)}$
$0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, \ldots$	$[n \text{ odd}](-1)^{\frac{n-1}{2}}/n!$	$\sin z$
$1, 0, -\frac{1}{2!}, 0, \frac{1}{4!}, 0, \ldots$	$[n \text{ even}](-1)^{\frac{n}{2}}/n!$	$\cos z$

- $H_n = \sum_{i=1}^{n} \frac{1}{i}$ (harmonic numbers)
- F_n — Fibonacci numbers

EGFs of certain sequences

$$
\begin{array}{l|l}
a_n = 1 & \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} = \mathrm{e}^z \\
a_{m,n} = \left\{ {n \atop m} \right\} & \sum_{m,n=0}^{\infty} a_{m,n} \frac{z^n}{n!} w^m = \mathrm{e}^{w(\mathrm{e}^z - 1)} \\
a_{m,n} = \binom{n}{m} & \sum_{m,n=0}^{\infty} a_{m,n} \frac{z^n}{n!} w^m = \mathrm{e}^{z + wz} \\
a_{m,n} = \left[{n \atop m} \right] & \sum_{m,n=0}^{\infty} a_{m,n} \frac{z^n}{n!} w^m = \frac{1}{(1-z)^w}
\end{array}
$$

- $\left[{n \atop m} \right]$ — number of permutations of n elements with m permutation cycles (see next lecture)
    - Stirling numbers of first kind

Discrete Mathematics, 12th lecture

Theory of Counting

Peeter Laud

Cybernetica AS

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Counting exercises

- In how many ways can a stripe of cloth with n stripes be colored with k different colors?
  - Turning around the cloth will not change the pattern.
- In how many ways can a n-bead necklace be made from beads of k different colors?
- In how many ways can the corners of a cube be colored so, that 3 corners are red, 3 are green, 2 are blue?

General task

- There are sets X and C. We are counting functions $f : X \to C$ in a certain manner.
  - Let $m = |X|$, $n = |C|$.
  - Let $\mathcal{F}$ be the set of functions from X to C.
- We have a set of permutations $\mathcal{G} = \{\pi_1, \pi_2, \ldots, \pi_k\} \subseteq S_X$.
  - $f, g : X \to C$ are equivalent if $\exists i : f \circ \pi_i = g$.
  - We count equivalence classes of functions, not functions themselves.
- Let us call the functions $f : X \to C$ the colorings of X.
  - Their number depends on the structure of $\mathcal{G}$ and on the number of colors.
    - $\mathcal{G}$ determines the size of X.
  - Let $t_{n,\mathcal{G}}$ denote the number of colorings.
- Later we also handle the case where the number of uses of each color has been given.

Groups and subgroups of permutations

- The set S_X of all permutations of X is a group wrt. the composition operation $\circ$:
  - S_X is closed wrt. $\circ$: if $\pi_1, \pi_2 \in S_X$, then $\pi_2 \circ \pi_1 \in S_X$;
  - $\circ$ is associative, there is unit element, each element has an inverse.
- A subset $\mathcal{H} \subseteq S_X$ is a subgroup of S_X if
  - $\mathcal{H}$ is closed wrt. $\circ$;
  - the identity permutation belongs to $\mathcal{H}$;
  - $\mathcal{H}$ is closed wrt. taking inverses.

  Denote $\mathcal{H} \leqslant S_X$. The set $\mathcal{H}$ is then also a group wrt. $\circ$

The set of permutations $\mathcal{G}$ in our task must be a subgroup of S_X.

Equivalences generated by permutation groups

- Let X be a set and $\mathcal{G} \leqslant S_X$.
- Define an equivalence $\sim_{\mathcal{G}}$ on X as follows:

$$x_1 \sim_{\mathcal{G}} x_2 \Leftrightarrow \exists \pi \in \mathcal{G} : \pi(x_1) = x_2 \ .$$

- **Lemma.** $\sim_{\mathcal{G}}$ is an equivalence relation.
- The equivalence classes of $\sim_{\mathcal{G}}$ are called orbits.
    - Let $\langle x \rangle$ denote the orbit of x — the equivalence class $x / \sim_{\mathcal{G}}$.
- The set of all equivalence classes of $\sim_{\mathcal{G}}$ is denoted $X/\mathcal{G}$.

Size of orbits

Definition

$$\mathrm{fix}(\pi) = \{x \in X \mid \pi(x) = x\} \qquad \text{(fixed points of } \pi \in \mathcal{G}\text{)}$$

$$\mathcal{G}_x = \{\pi \in \mathcal{G} \mid \pi(x) = x\} \qquad \text{(stabilizers of } x \in X\text{)}$$

Lemma

$$|\langle x \rangle| = \frac{|\mathcal{G}|}{|\mathcal{G}_x|}$$

Proof.

- Let $y \in \langle x \rangle$.
- Let $\mathcal{G}_{x \to y} = \{\pi \in \mathcal{G} \mid \pi(x) = y\}$. Let ξ be a fixed element of $\mathcal{G}_{x \to y}$.
- $\pi \mapsto \xi \circ \pi$ is a bijection from $\mathcal{G}_x$ to $\mathcal{G}_{x \to y}$.

Hence for each $y \in \langle x \rangle$, there are $|\mathcal{G}_x|$ elements of $\mathcal{G}$ mapping x to y. □

Number of orbits

Lemma (Burnside)

$$|X/\mathcal{G}| = \frac{1}{|\mathcal{G}|}\sum_{\pi\in\mathcal{G}} |\mathrm{fix}(\pi)|$$

Proof.

$$\sum_{\pi\in\mathcal{G}} |\mathrm{fix}(\pi)| = \sum_{x\in X} |\mathcal{G}_x| = |\mathcal{G}|\sum_{x\in X}\frac{1}{|\langle x\rangle|} = |\mathcal{G}|\sum_{\langle y\rangle\in X/\mathcal{G}}\sum_{x\in\langle y\rangle}\frac{1}{|\langle x\rangle|} =$$

$$|\mathcal{G}|\sum_{\langle y\rangle\in X/\mathcal{G}}\sum_{x\in\langle y\rangle}\frac{1}{|\langle y\rangle|} = |\mathcal{G}|\sum_{\langle y\rangle\in X/\mathcal{G}} 1 = |\mathcal{G}|\cdot|X/\mathcal{G}|$$

□

Permutations acting on colorings

- Let $\pi \in \mathcal{G} \leqslant S_X$. Let $f : X \to C$.
- Define the action (*toime*) of π on f: $\pi f = f \circ \pi^{-1}$.
  - Let $\widetilde{\pi}(f) = \pi f$. Let $\widetilde{\mathcal{G}} = \{\widetilde{\pi} \,|\, \pi \in \mathcal{G}\}$.
  - **Lemma.** $\widetilde{\pi} \in S_{\mathcal{F}}$.
- **Lemma.**
  - $(\pi' \circ \pi)f = \pi'(\pi f)$ and
  - $\pi f = f' \Leftrightarrow (\pi^{-1})f' = f$

  for any f, f', π, π'.
- **Corollary.** $\widetilde{\mathcal{G}} \leqslant S_{\mathcal{F}}$.
- Each orbit of $\widetilde{\mathcal{G}}$ is a coloring that is distinguishable from other colorings.
- $t_{n,\mathcal{G}}$ is equal to the number of orbits of $\widetilde{\mathcal{G}}$.
  - We could compute it using Burnside's lemma
  - What are the fixed points of $\widetilde{\pi}$? How many are there?

Permutation cycles

- Sequence $(x_1, x_2, \ldots, x_r) \in X^r$ is a permutation cycle of $\pi \in S_X$ if $\pi(x_1) = x_2, \pi(x_2) = x_3, \ldots, \pi(x_{r-1}) = x_r, \pi(x_r) = x_1$.
- Each permutation can be expressed as a "product" of its permutation cycles. E.g. π:

x	1	2	3	4	5	6	7	8	9	10	11	12
$\pi(x)$	4	6	3	7	10	2	1	9	5	12	11	8

  can be written (1 4 7)(2 6)(3)(5 10 12 8 9)(11)
  - Cycles of length 1 are often omitted in the write-up
- This write-up is unique up to cyclic shifts of each cycle and permutation of cycles.
- Let $c(\pi)$ be the number of cycles of π.
- Let $c_i(\pi)$ be the number of cycles of π of length i.
- The number of permutations of n elements with m cycles is $\left[{n \atop m}\right]$.

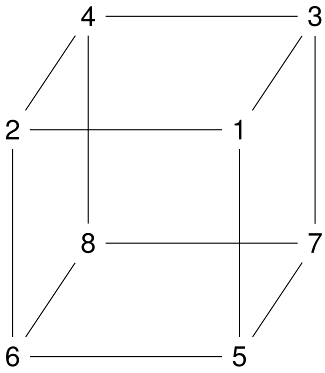
Two elements:

- The identity permutation.
  - $(1)(2)\cdots(n)$
  - n cycles of length 1.
- Turing the stripe over.
  - $(1\ n)(2\ n-1)(3\ n-2)\cdots$
  - If n even: $n/2$ cycles of length 2.
  - If n odd: one cycle of length 1 and $(n-1)/2$ cycles of length 2.

$2n$ elements.

- Rotation by k positions ($0 \leqslant k \leqslant n-1$)
    - Let $d = \gcd(n, k)$.
    - d cycles of length n/d.
- Change of direction followed by rotation by k positions.
    - n odd: 1 cycle of length 1 and $(n-1)/2$ cycles of length 2.
    - k and n even: 2 cycles of length 1 and $(n-2)/2$ cycles of length 2.
    - k odd, n even: $n/2$ cycles of length 2.

- *id*
- (2 3 5)(4 7 6)
- (1 4 7)(2 8 5)
- (1 6 4)(3 5 8)
- (1 6 7)(2 8 3)
- (2 5 3)(4 6 7)
- (1 7 4)(2 5 8)
- (1 4 6)(3 8 5)
- (1 7 6)(2 3 8)
- (1 2 4 3)(5 6 8 7)
- (1 5 7 3)(2 6 8 4)
- (1 5 6 2)(3 7 8 4)
- (1 4)(2 3)(5 8)(6 7)
- (1 7)(2 8)(3 5)(4 6)
- (1 6)(2 5)(3 8)(4 7)
- (1 8)(2 6)(3 7)(4 5)
- (1 8)(2 7)(3 4)(5 6)
- (1 8)(2 4)(3 6)(5 7)
- (1 5)(2 7)(3 6)(4 8)
- (1 2)(3 6)(4 5)(7 8)
- (1 3)(2 7)(4 5)(6 8)
- (1 3 4 2)(5 7 8 6)
- (1 3 7 5)(2 4 8 6)
- (1 2 6 5)(3 4 8 7)



Fixed points of permutations on colorings

Lemma

$f \in \operatorname{fix}(\widetilde{\pi})$ iff f is constant on each cycle of π.

Hence $|\operatorname{fix}(\widetilde{\pi})| = n^{c(\pi)}$.

Theorem

The number of colorings of X is

$$t_{n,\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{\pi \in \mathcal{G}} n^{c(\pi)} \quad .$$

Weighed colorings

- Let the weight of the color $i \in C$ be $w(i) = z_i$.
- Let the weight of the coloring $f : X \to C$ be $w(f) = \prod_{x \in X} w(f(x))$.
- **Lemma.** If $f \circ \pi = g$, then $w(f) = w(g)$. Thus we can write $w(\langle f \rangle)$.
- The pattern inventory is the polynomial

$$W_{\mathcal{G}}(z_1, \ldots, z_n) = \sum_{\langle f \rangle \in \mathcal{F}/\sim_{\mathcal{G}}} w(\langle f \rangle) \ .$$

$W_{\mathcal{G}}(1, \ldots, 1) = t_{n,\mathcal{G}}$.

Coefficient of a monomial in pattern inventory

- Let $\mathbf{k} = (k_1, \ldots, k_n)$, where $k_1 + \cdots + k_n = m$.
    - Let $\mathbf{z}^{\mathbf{k}}$ denote $z_1^{k_1} \cdots z_n^{k_n}$.
- Let $\mathcal{F}_{\mathbf{k}} = \{f \in \mathcal{F} \,|\, w(f) = \mathbf{z}^{\mathbf{k}}\}$.
- Any $\widetilde{\pi} \in \widetilde{\mathcal{G}}$ is a permutation on $\mathcal{F}_{\mathbf{k}}$.
    - Let $\mathrm{fix}_{\mathbf{k}}(\widetilde{\pi}) = \mathrm{fix}(\widetilde{\pi}) \cap \mathcal{F}_{\mathbf{k}}$.

From the Burnside lemma:

$$\begin{aligned} W_{\mathcal{G}}(z_1, \ldots, z_n) &= \sum_{\sum \mathbf{k} = m} \frac{1}{|\mathcal{G}|} \sum_{\pi \in \mathcal{G}} |\mathrm{fix}_{\mathbf{k}}(\widetilde{\pi})| \mathbf{z}^{\mathbf{k}} \\ &= \frac{1}{|\mathcal{G}|} \sum_{\pi \in \mathcal{G}} \sum_{\sum \mathbf{k} = m} |\mathrm{fix}_{\mathbf{k}}(\widetilde{\pi})| \mathbf{z}^{\mathbf{k}} \end{aligned}$$

- Let $\mathbf{k} = (k_1, \ldots, k_n)$, where $k_1 + \cdots + k_n = m$.
  - Let $\mathbf{z}^{\mathbf{k}}$ denote $z_1^{k_1} \cdots z_n^{k_n}$.
- Let $\mathcal{F}_{\mathbf{k}} = \{f \in \mathcal{F} \mid w(f) = \mathbf{z}^{\mathbf{k}}\}$.
- Any $\widetilde{\pi} \in \widetilde{\mathcal{G}}$ is a permutation on $\mathcal{F}_{\mathbf{k}}$.
  - Let $\mathrm{fix}_{\mathbf{k}}(\widetilde{\pi}) = \mathrm{fix}(\widetilde{\pi}) \cap \mathcal{F}_{\mathbf{k}}$.

From the Burnside lemma:

$$\begin{aligned} W_{\mathcal{G}}(z_1, \ldots, z_n) &= \sum_{\sum \mathbf{k} = m} \frac{1}{|\mathcal{G}|} \sum_{\pi \in \mathcal{G}} |\mathrm{fix}_{\mathbf{k}}(\widetilde{\pi})| \mathbf{z}^{\mathbf{k}} \\ &= \frac{1}{|\mathcal{G}|} \sum_{\pi \in \mathcal{G}} \sum_{\sum \mathbf{k} = m} |\mathrm{fix}_{\mathbf{k}}(\widetilde{\pi})| \mathbf{z}^{\mathbf{k}} \end{aligned}$$

Coefficients of monomials for a single permutation

- What is $\mathcal{S}_\pi = \sum_{\sum \mathbf{k}=m} |\mathrm{fix}_{\mathbf{k}}(\widetilde{\pi})|\mathbf{z}^{\mathbf{k}}$?
- Let $f \in \mathcal{F}_{\mathbf{k}}$. Then $f \in \mathrm{fix}_{\mathbf{k}}(\widetilde{\pi})$, iff f is a constant on each cycle of π.
- Let us sum over each cycle of π, assigning one of the colors $1, \ldots, n$ to each of them.
  - $i_{r,s}$ will be the variable storing the color of the r-th cycle of length s.

$$\mathcal{S}_\pi = \sum_{i_{1,1}=1}^{n} \cdots \sum_{i_{1,c_1(\pi)}=1}^{n} \sum_{i_{2,1}=1}^{n} \cdots \sum_{i_{2,c_2(\pi)}=1}^{n} \cdots \sum_{i_{m,1}=1}^{n} \cdots \sum_{i_{m,c_m(\pi)}=1}^{n} z_{i_{1,1}} \cdots z_{i_{1,c_1(\pi)}} z_{i_{2,1}}^2 \cdots z_{i_{2,c_2(\pi)}}^2 \cdots z_{i_{m,1}}^m \cdots z_{i_{m,c_m(\pi)}}^m$$

- Let $M_{n,s} = z_1^s + z_2^s + \cdots + z_n^s$.

$$\mathcal{S}_\pi = M_{n,1}^{c_1(\pi)} \cdot M_{n,2}^{c_2(\pi)} \cdots M_{n,m}^{c_m(\pi)}$$

The pattern inventory

We've just proved

Theorem (Polya)

$$W_{\mathcal{G}}(z_1,\ldots,z_n) = \frac{1}{|\mathcal{G}|}\sum_{\pi\in\mathcal{G}} M_{n,1}^{c_1(\pi)}\cdots M_{n,m}^{c_m(\pi)}$$

The cycle index polynomial

Definition

The cycle index polynomial (*tsüklilisuse indikaator*) of $\mathcal{G} \leqslant S_X$ is

$$Z_{\mathcal{G}}(w_1, \ldots, w_m) = \frac{1}{|\mathcal{G}|} \sum_{\pi \in \mathcal{G}} w_1^{c_1(\pi)} \cdots w_m^{c_m(\pi)}$$

Corollary

$$W_{\mathcal{G}}(z_1, \ldots, z_n) = Z_{\mathcal{G}}(M_{n,1}, \ldots, M_{n,m})$$