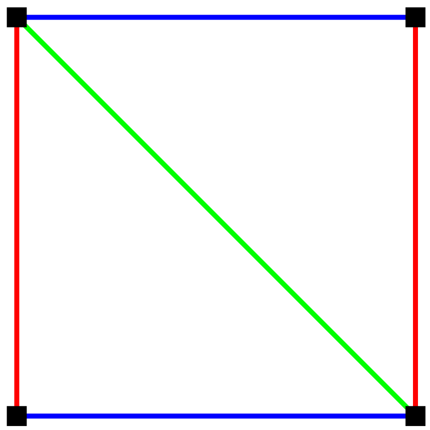
Coloring edges

Let $G = (V, E)$ be a graph without loops. Its *(correct) edge coloring with k colors* is a function $\gamma : E \longrightarrow \{1, \ldots, k\}$ such that

- for any two different edges $e_1, e_2 \in E$ with a common endpoint we have $\gamma(e_1) \neq \gamma(e_2)$.

Stating it otherwise, all the edges incident with some vertex must be colored differently.



Example: let us consoder a set of (school) classes X and a set of teachers Y. For each class it is known how many lessons a given teacher must teach to this class.

The task is to compose a time-table for the school.

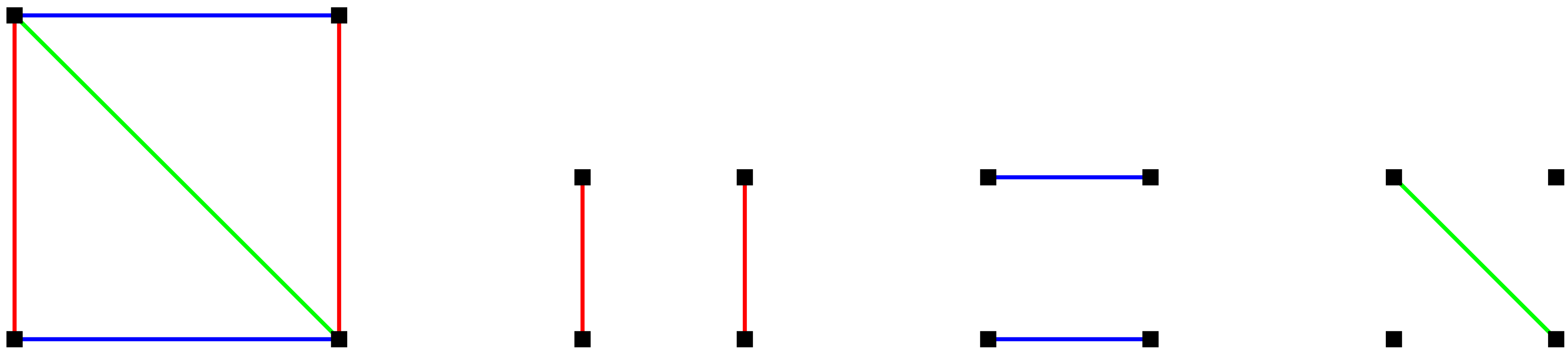
Consider a bipartite graph with vertex set $X \cup Y$ and the number of edges between $x \in X$ and $y \in Y$ showing, how many lessons the teacher y teaches to the class x.

The time-table can be represented as a correct edge coloring, where the edge colors are possible time slots for the lessons.

Let $G = (V, E)$ be a graph, γ its edge coloring and i one of the colors.

The set $\{e \mid e \in E, \gamma(e) = i\}$ is a matching.

Coloring the edge set can be thought of as partitioning this set into matchings.



Let $G = (V, E)$ be a graph. Assume it has a correct edge coloring with k colors, but no correct edge coloring with $k - 1$ colors.

The number k is called *edge chromatic number* and denoted $\chi'(G)$.

Let $\Delta(G) = \max_{v \in V} \deg(v)$ be the maxmal vertex degree of graph G.

Obviously, $\chi'(G) \geq \Delta(G)$.

An example where $\chi'(G) > \Delta(G)$: odd cycles.

Theorem. In a bipartite graph G we have $\chi'(G) = \Delta(G)$.

Proof. First turn G into a $\Delta(G)$-regular graph.

1. If one of the vertex set parts has less vertices than the other, then add the missing number of vertices to make the parts equal.

2. If some of the vertices in one part has a degree less than $\Delta(G)$, then a similar vertex must also exist in the other part. Join these two by an edge.

If the edges of the new graph can be colored using $\Delta(G)$ colors, the same holds true for the original graph as well.

Thus we can consider only k-regular bipartite graphs G.

1. k-regular bipartite graph G has a complete matching M_1.
2. Remove the edges of M_1. The remaining graph is a $(k-1)$-regular bipartite graph.
3. This graph has a complete matching M_2.
4. Remove the edges of M_2. The remaining graph is a $(k-2)$-regular bipartite graph.
5. etc.

This way we partition the edge set of G into k perfect matchings $M_1, \ldots, M_k$. These matchings give a suitable coloring. $\square$

Theorem (Vizing). Let $G = (V, E)$ be a <u>simple</u> graph. Then $\chi'(G) \leq \Delta(G) + 1$.

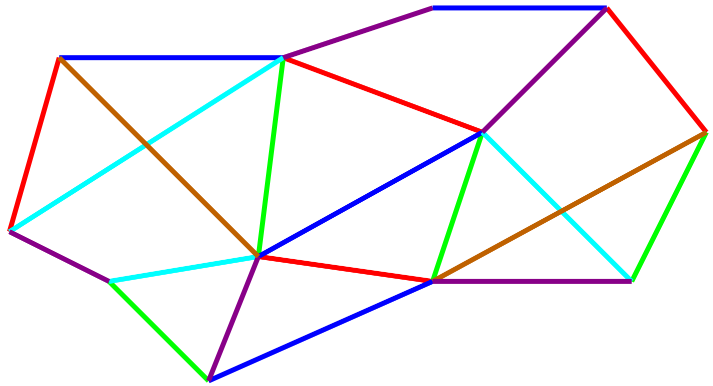
Proof is by induction over the number of vertices. The claim is obvious if $|V| = 1$.

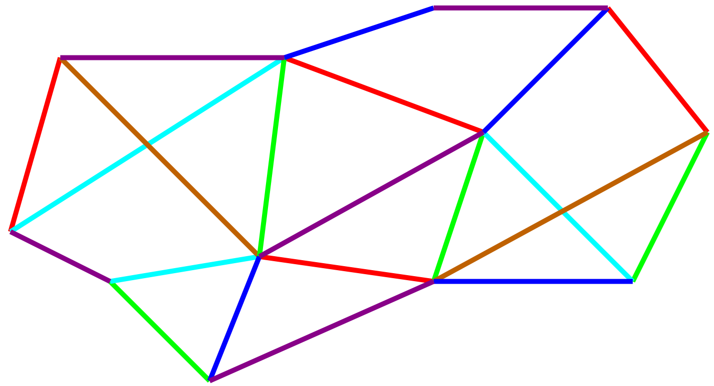
We have to show the following:

> Let $G = (V, E)$ be a simple graph and let $k = \Delta(G) + 1$. Choose a vertex $v \in V$ in graph G and let the edges of $G \backslash v$ be colorable with k colors. Then the edges of G can also be colored with k colors.

We will prove this statement using induction over k. In fact, we will even prove a slightly stronger result.

Lemma. Let $G = (V, E)$ be a graph and γ its edge coloring. Let $E' \subseteq E$ be an edge subset being colored with some two colors. Consider the graph $G' = (V, E')$.



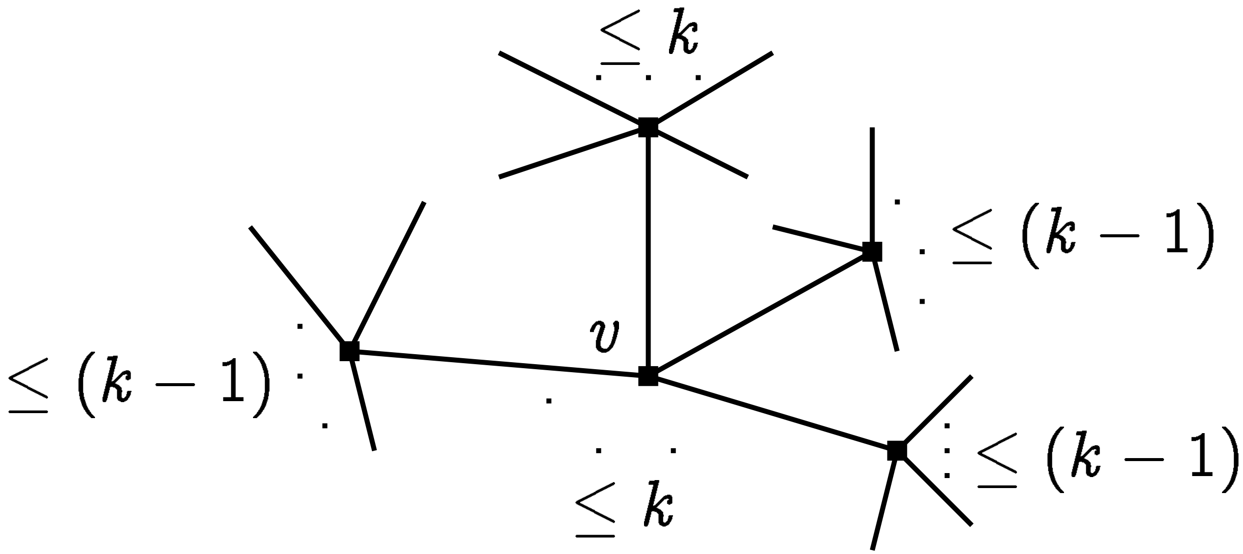


Let H be a connected component of graph G'. If we exchange the colors of the edges of H, we again get a correct coloring of the edges of G

Proof is pretty straightforward. $\square$

Lemma. Let $G = (V, E)$ be a simple graph and $k \in \mathbb{N}$. Let $v \in V$ be such that

- $\deg(v) \le k$. If $w \in V$ is the neighbour of v, then $\deg(w) \le k$.
- Vertex v has at most one neighbour with degree k.



Let the edges of $G \backslash v$ be colorable with k colors. Then the edges of k are colorable with k colors.

Proof by induction on k.

Base. $k = 1$.

Thus $\deg(v) = 0$ or $\deg(v) = 1$.

If $\deg(v) = 0$, the edges of G coincide with the edges of $G \backslash v$.

If $\deg(v) = 1$, then let u be the neighbour of v. According to the assumption of the Lemma we have $\deg(u) \leq 1$, thus u — v is a connected component of G.

The coloring of G can be obtained from the coloring of $G \backslash v$ bu coloring the edge between u and v using the only available color.

Step. $k > 1$.

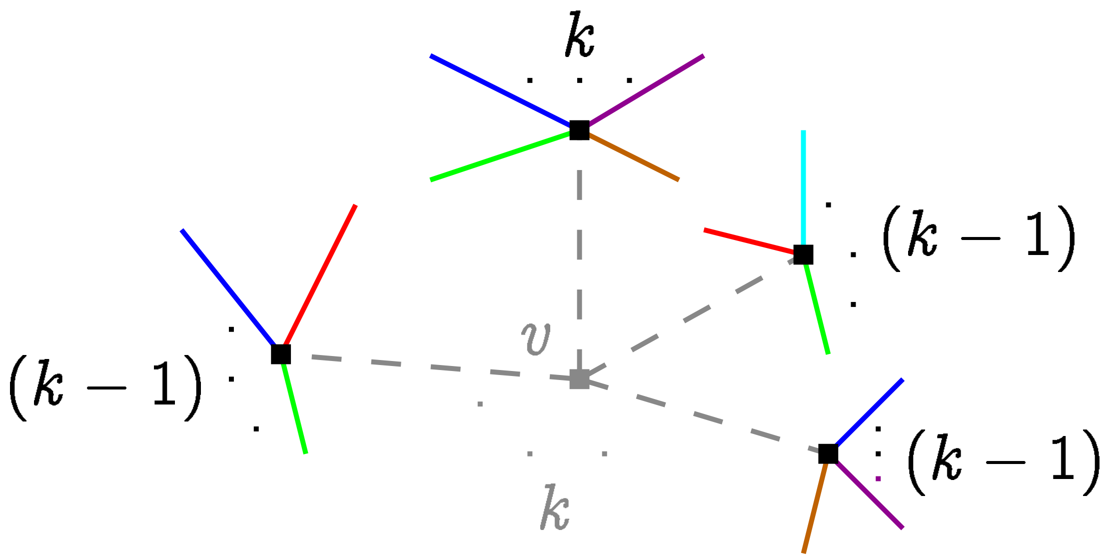
As long as $\deg(v) < k$, we add another vertex u and an edge $u — v$ to the graph G.

As long as the degree of some neighbour v' of v is less than k or $k-1$, we add another vertex u and an edge $u — v'$ to the graph G.

Thus we get a graph G with equalities holding in all the inequalities in the statement of the Lemma.

The modified graph is colorable with k colors iff the original graph was.

Let γ be the coloring of the graph $G\backslash v$ with k colors.



Consider the neighbours of the (removed) vertex v. For each $i \in \{1, \ldots, k\}$ let X_i be the set of such neighbours that have no incident edges colored with color i.

One of these vertices belongs to exactly one of the sets X_i, all the others belong to exactly two of these. Thus $\sum_{i=1}^{k} |X_i| = 2k - 1$.

We will be looking for γ such that there is a color i with $|X_i| = 1$.

That is, the edges colored with i are incident with all the neighbours of v, except for one.

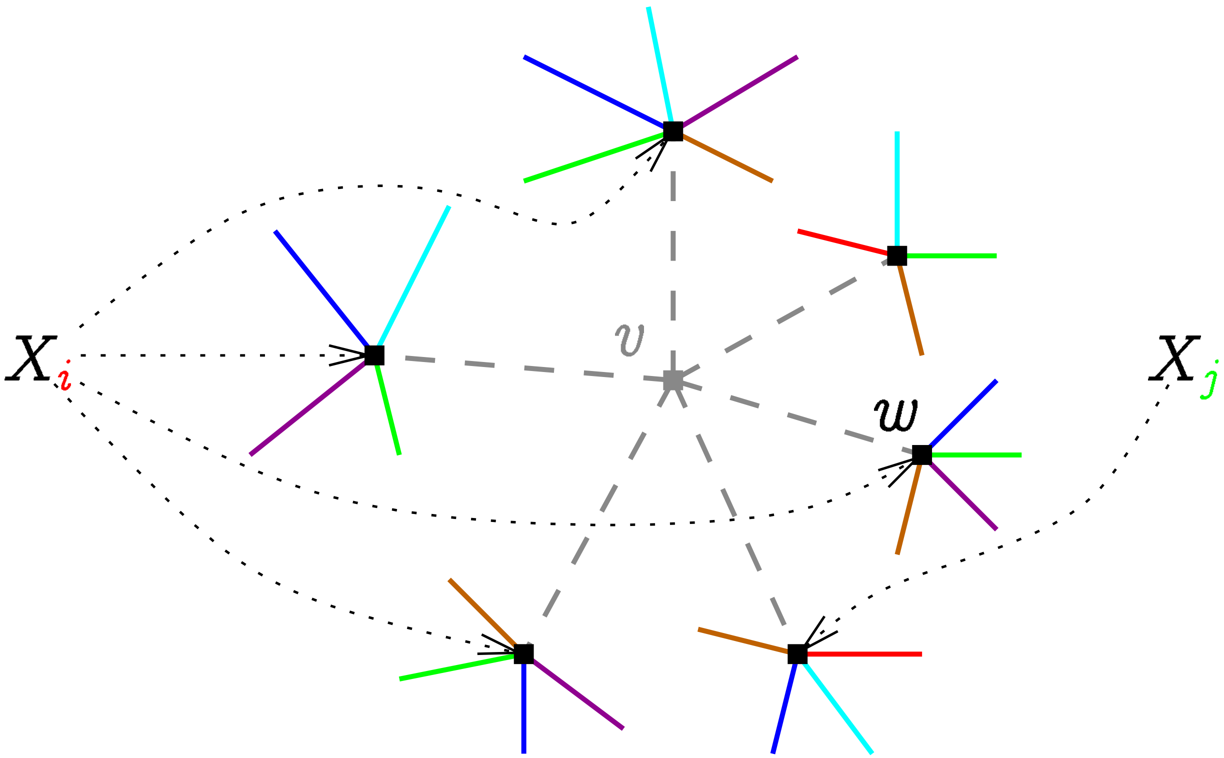
We will first show that we can choose γ so that for every $i, j \in \{1, \ldots, k\}$ we have $\big||X_i| - |X_j|\big| \leq 2$.

To do that we will prove that if for some i, j we have $|X_i| - |X_j| \geq 3$, then there is a coloring γ' such that $|X_i|$ is decreased by 1 and $|X_j|$ is increased by 1.

We will also prove that after a finite number of such steps $(\gamma \to \gamma')$ there will be no such i and j.

Let i and j be such that $|X_i| - |X_j| \geq 3$. Let $w \in X_i \backslash X_j$.

I.e., the number of vertices having an incident edge of color j is larger by at least 3 than the number of vertices having an incident edge of color i.

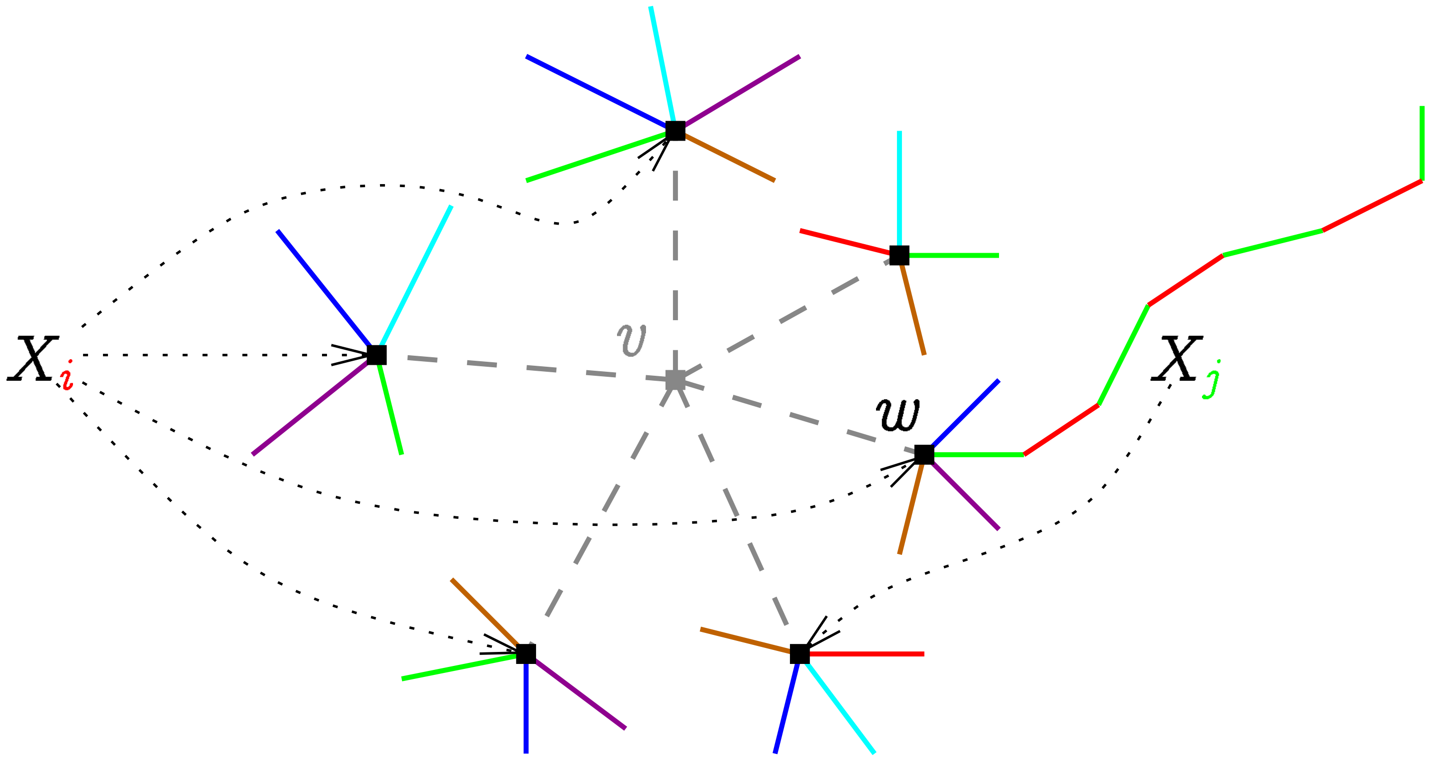


Let $E' \in E$ be the set of all edges e such that $\gamma(e) = i$ or $\gamma(e) = j$. Consider the graph $G' = (V, E')$.

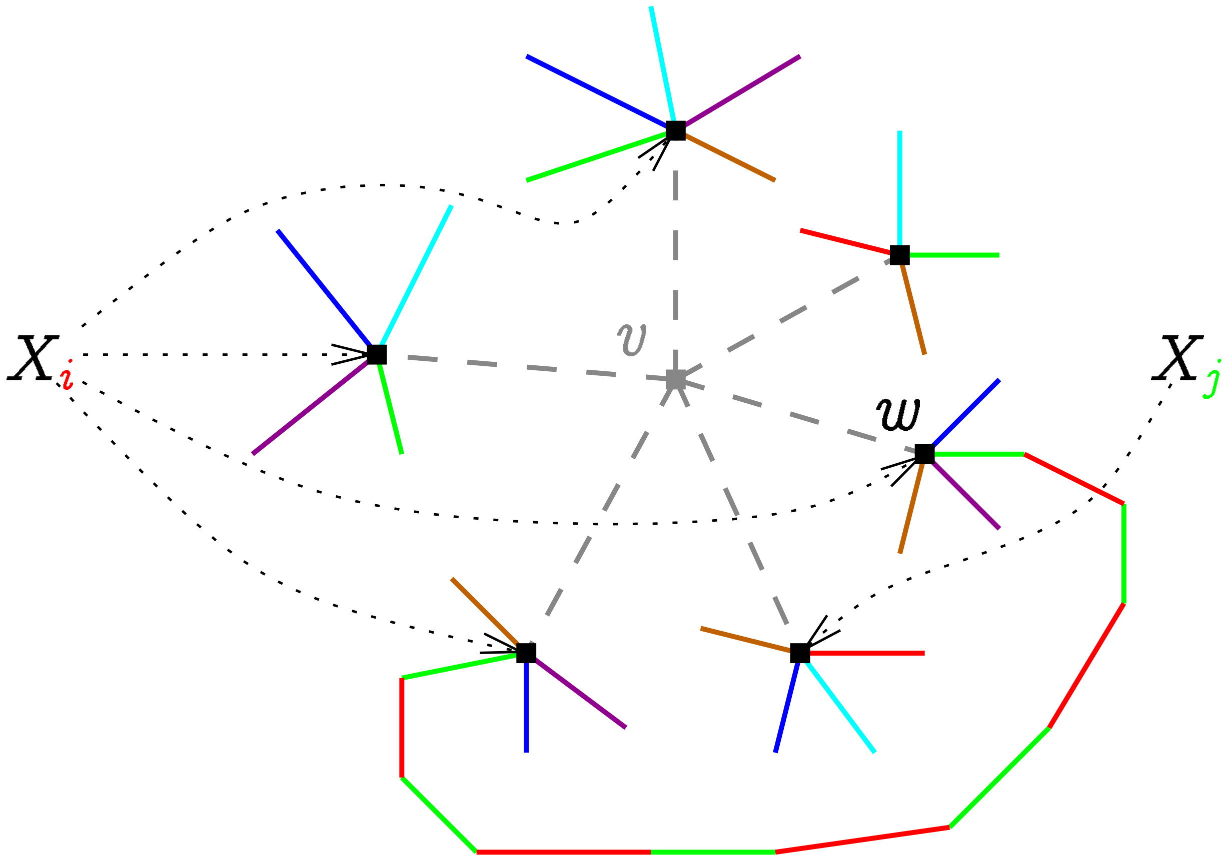
In G', all vertex degrees are ≤ 2. Thus the connected components of G' are isolated vertices, paths and cycles.

The vertex $w \in X_i \backslash X_j$ is an endpoint of some path.

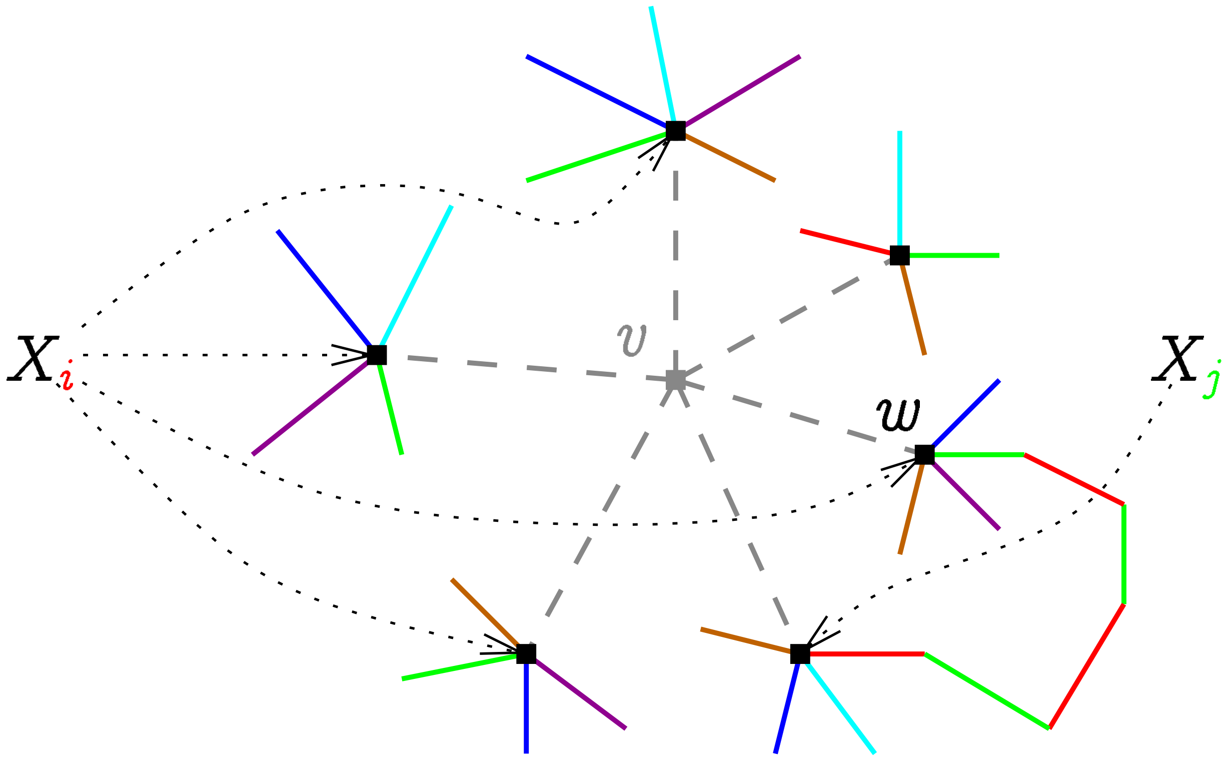
Where can the other endpoint be?



Somewhere else in graph G



In a vertex of the set $X_i \backslash X_j$

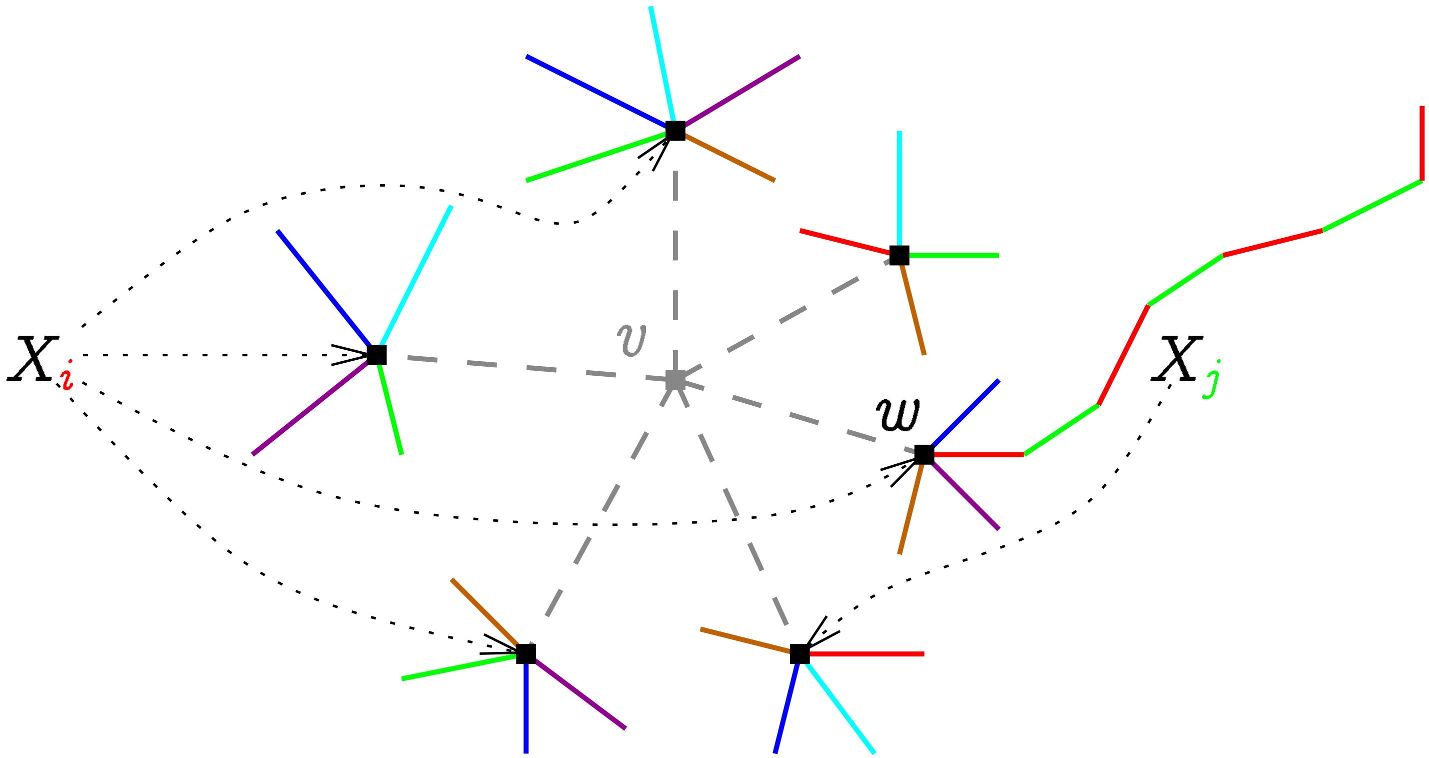


In a vertex of the set $X_j \setminus X_i$

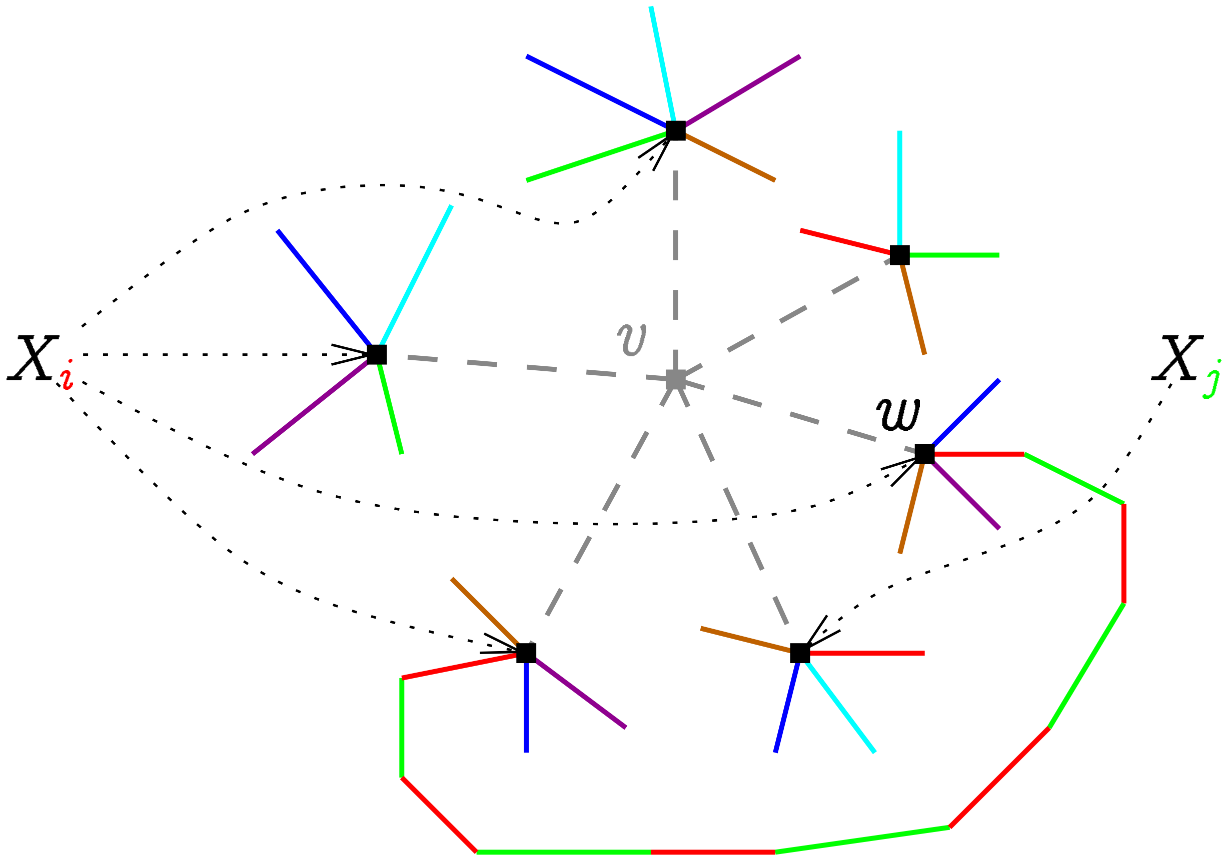
Since $|X_i| > |X_j|$, there exists $w \in X_i \backslash X_j$ such that the path that starts in it (being a connoected component in G') ends somewhere else than in the set $X_j \backslash X_i$.

In this path we exchange the colors i and j. We get a new coloring γ'.

$|X_i|$ and $|X_j|$ will change as follows:



$|X_i|$ decreases by one, $|X_j|$ increases by one



$|X_i|$ decreases by two, $|X_j|$ increases by two

To show the finiteness of the process, we need a *monovariant*, i.e. a quantity describing a coloring γ of the graph $G \backslash v$, such that

- On each step $(\gamma \to \gamma')$ it changes by a positive integer in a certain direction (e.g. decreases strictly).
- It has a fixed bound in this direction (e.g. 0).

A suitable quantity is $\sum_{i=1}^{k} |X_i|^2$.

Indeed, let $n_i, n_j \in \mathbb{N}$ such that $n_i - n_j \geq 3$. Then

$$(n_i - 1)^2 + (n_j + 1)^2 = {n_i}^2 + {n_j}^2 - 2(n_i - n_j) + 2 \leq {n_i}^2 + {n_j}^2 - 4$$
$$(n_i - 2)^2 + (n_j + 2)^2 = {n_i}^2 + {n_j}^2 - 4(n_i - n_j) + 8 \leq {n_i}^2 + {n_j}^2 - 4$$

We have shown that there is a coloring γ, such that the cardinalities of X_i differ by at most 2.

Average cardinality of the sets X_i is a bit less than 2 (namely $\frac{2k-1}{k}$). Thus the possible sets of cardinalities of X_i are $\{0,1,2\}$ and $\{1,2,3\}$.

If we have $\{1,2,3\}$, then there must exist i such that $|X_i| = 1$, otherwise the average cardinality is at least 2.

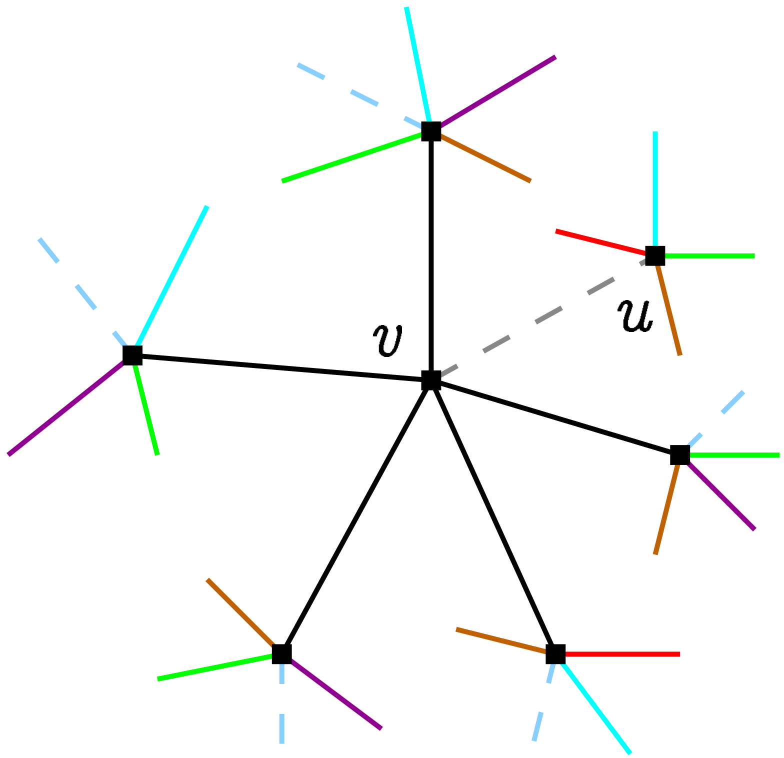
If we have $\{0,1,2\}$, then there must exist i such that $|X_i| = 1$, since the sum of cardinalities of X_i is odd $(2k-1)$.

W.l.o.g. assume that this i is k. Let $\{u\} = X_k$.

Let H be obtained from G by deleting

- all edges that γ colors with color k;
- the edge between v and u.

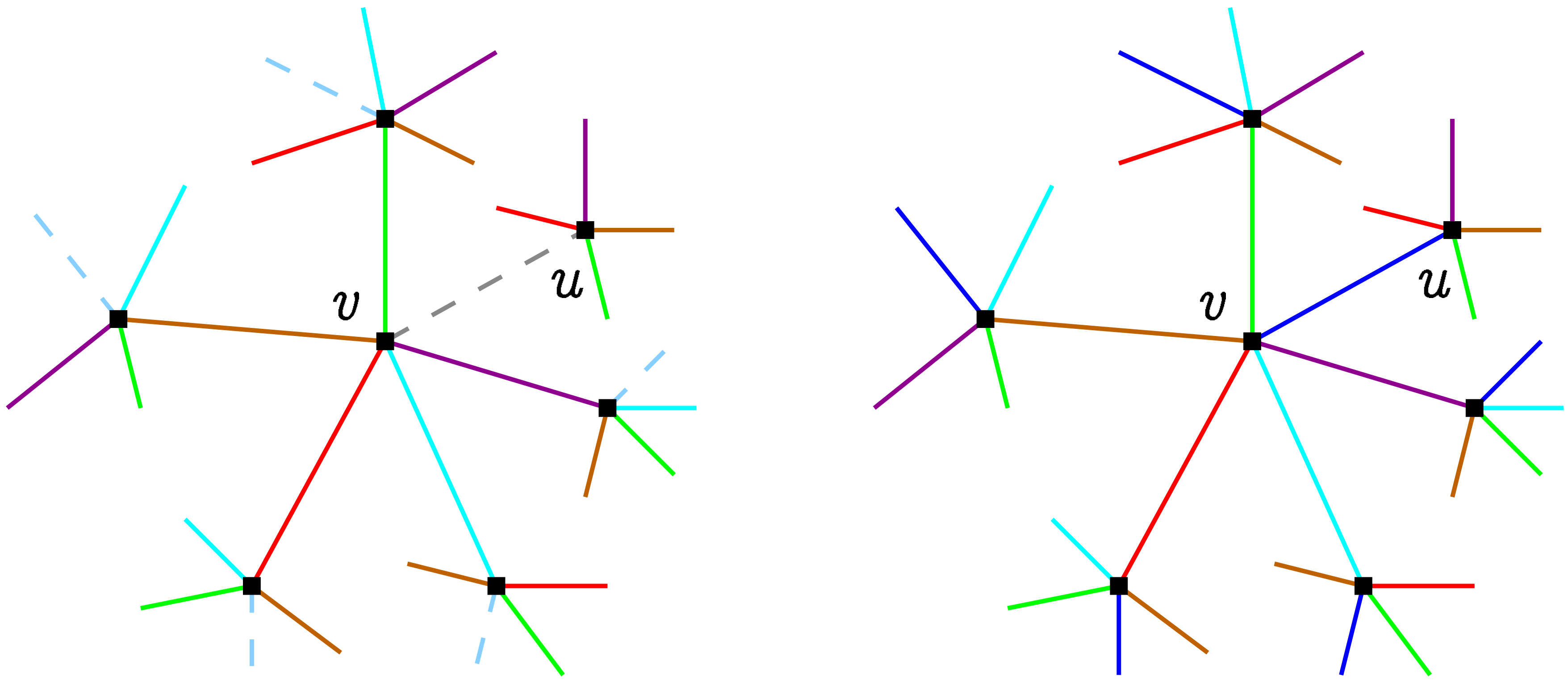
All the deleted edges form a matching in G.



Coloring γ without the color k is a coloring of the edges of $H \backslash v$ using $(k-1)$ colors.

The degree of v and its every neighbour (in H) has decreased by 1.

Induction hypothesis can be applied to graph H and vertex v. Thus the edges of H can be colored with $k-1$ colors. Let γ' be such a coloring.



We obtain the required coloring of G with k colors by coloring all the deleted edges with color k. $\square$

Let $\mu(G)$ be the maximum multiplicity of edges in the graph G.

Theorem. If G is without loops, then $\chi'(G) \leq \Delta(G) + \mu(G)$.

In the following we will distinguish between colorings and correct colorings.

For a coloring $\gamma : E \longrightarrow \{1, \ldots, k\}$ let $\tilde{\gamma}(v)$ be the number of colors that occur by the vertex $v \in V$.

A coloring γ with k colors is optimal if $\sum_{v \in V} \tilde{\gamma}(v)$ is the maximum possible among colorings with k colors.

Obviously, if there exist correct colorings with k colors, then exactly those are optimal.

Lemma 1. Let G be a connected graph that is not an odd cycle. There exist a coloring γ with 2 colors, such that $\tilde{\gamma}(v) = 2$ for any $v \in V$ where $\deg(v) \geq 2$.

Proof. First consider the case where G has an Eulerian walk C.

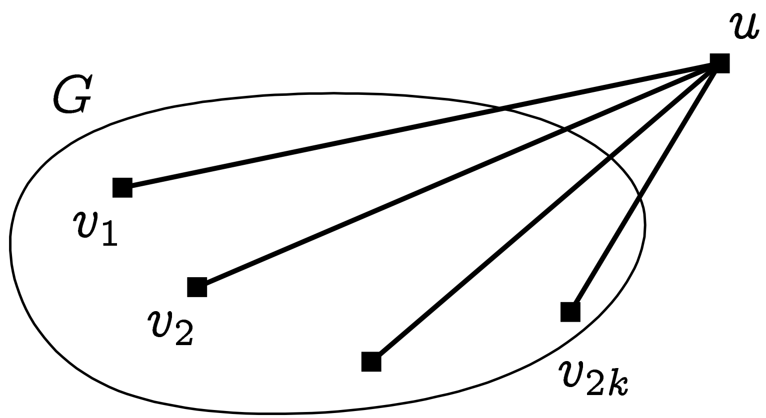
Move along C and color the edges in alternate colors.

If $|E|$ is odd then start from a vertex with degree ≥ 3.

If G is not Eulerian then make it Eulerian by

- introducing an extra vertex u;
- connecting u with all odd-degree vertices in G.
  - There is an even number of them.

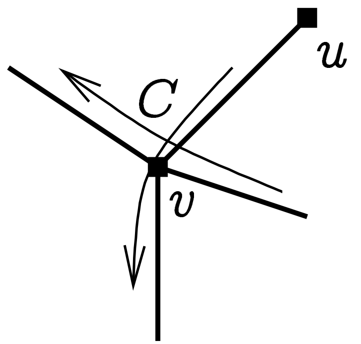
let G' be the resulting graph



Again consider Eulerian walk C and color the edges alternately along it.

If $\deg_G(v)$ is even then C enters it and leaves it along edges in G.

If $\deg_G(v) > 1$ is odd, then $\deg_G(v) \geq 3$ and $\deg_{G'}(v) \geq 4$.



At least once, C enters v and then leaves it along edges in G. $\square$.

Lemma 2. Let γ be an optimal coloring of $G = (V, E)$ with k colors. Let i, j be two colors. Let $E' = \gamma^{-1}(\{i, j\})$. Consider the graph $G' = (V, E')$.

Let $v \in V$ be such that

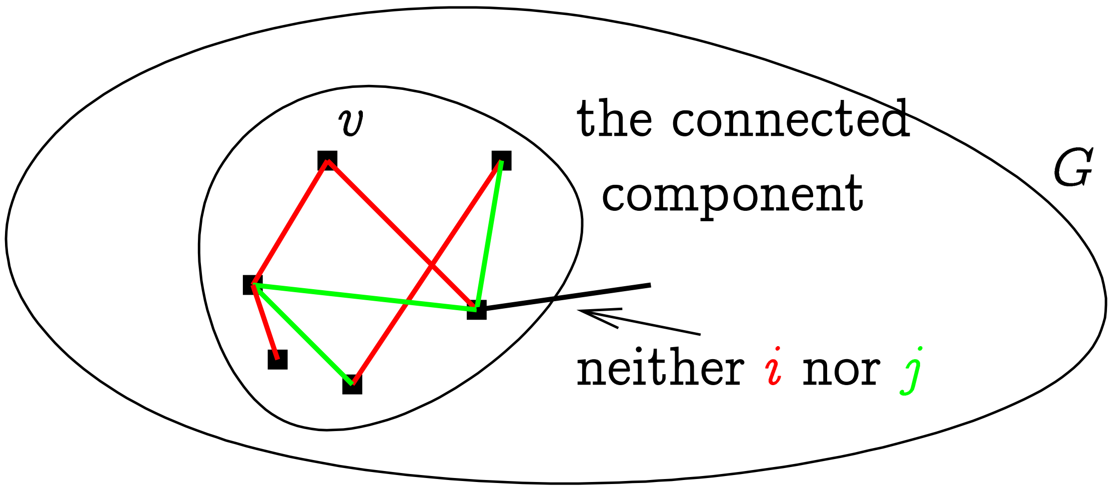
- $\deg_{\gamma^{-1}(i)}(v) \geq 2$;
- $\deg_{\gamma^{-1}(j)}(v) \geq 0$.

Then the connected component of G' containing v is an odd-length cycle.

Proof. Take this connected component. Recolor it using the previous lemma.

- Assuming it wasn't an odd-length cycle.

Then $\tilde{\gamma}(v)$ increases and $\tilde{\gamma}(\cdot)$ does not decrease for other vertices. Hence γ was not optimal.



The previous lemma isn't applicable only if this connected component is an odd-length cycle. ∎

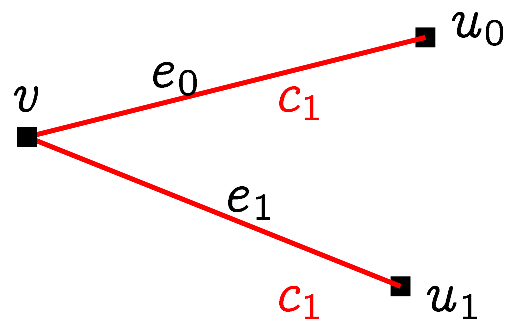
Lemma 3. Let γ be an optimal coloring of $G = (V, E)$ with k colors. Let e_1, e_2 be (a part of) a multiple edge between u and v. If $\deg(u) \leq k$ then $\gamma(e_1) \neq \gamma(e_2)$.

Proof. If $\gamma(e_1) = \gamma(e_2)$ then recolor e_2 with a color not occurring at u. This increases $\tilde{\gamma}(u)$ and does not decrease $\tilde{\gamma}(v)$. Hence γ was not optimal. $\square$

Proof of theorem. Let γ be an optimal coloring of $G = (V, E)$ with $\Delta(G) + \mu(G)$ colors. Assume that γ is not correct.

Let v be a vertex where a color c_1 occurs at least twice. Let $e_0, e_1 \in E$ be incident with v, such that $\gamma(e_0) = \gamma(e_1) = c_1$.

Let u_0, u_1 be the other end vertices of e_0, e_1. By the previous lemma, $u_0 \neq u_1$.


u_0
e_0
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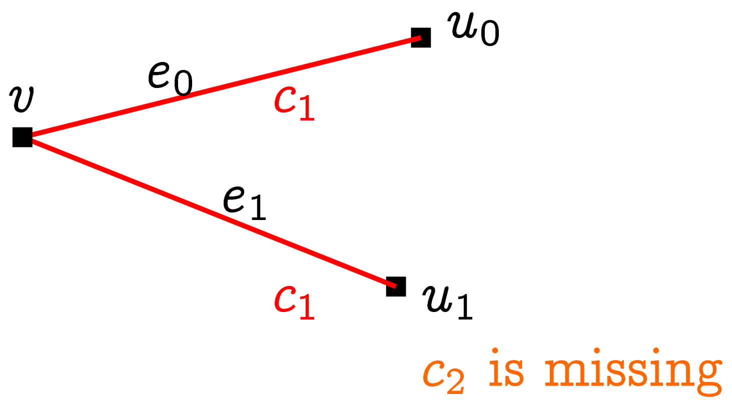
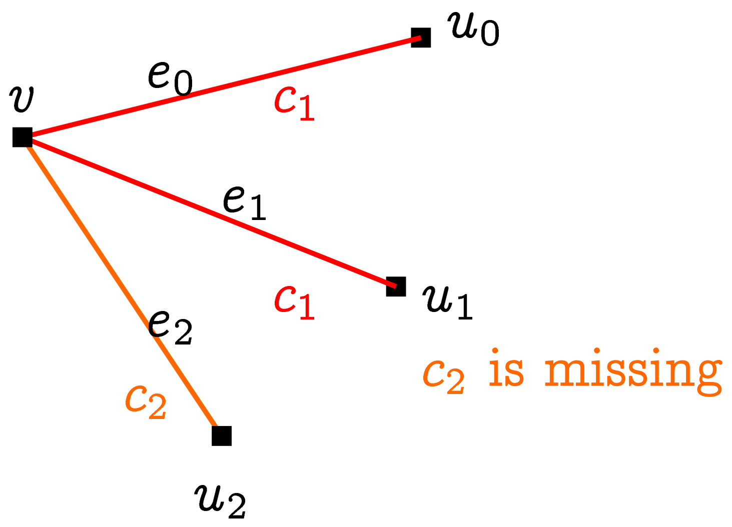
u_1



Let color c_2 be missing at u_1

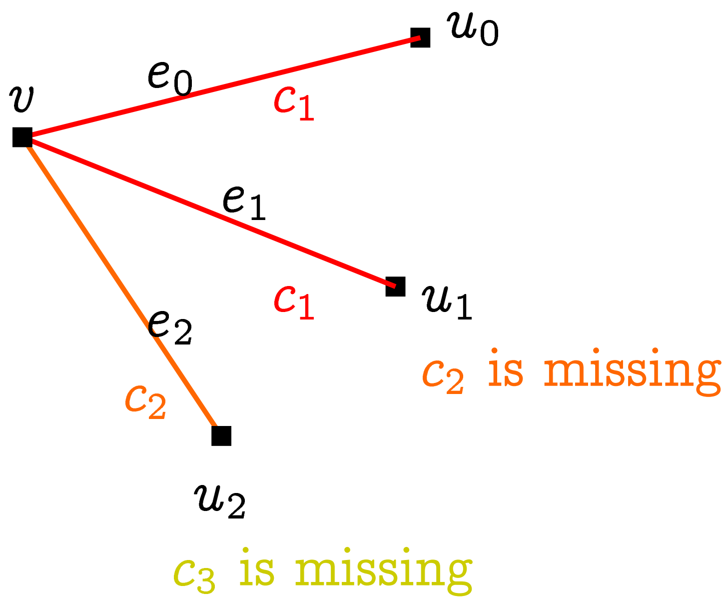


Let color c_2 be missing at u_1

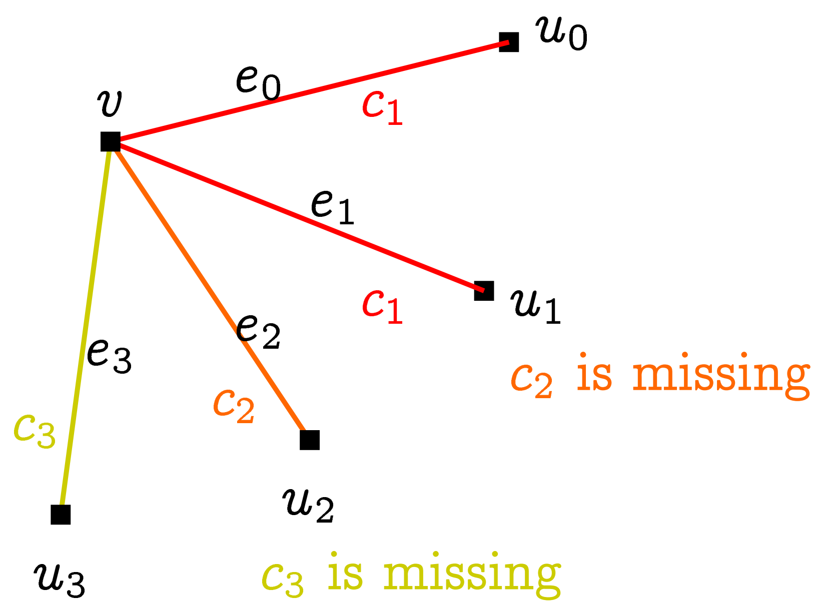
Let e_2 be incident to v, such that $\gamma(e_2) = c_2$

Let u_2 be the other end-vertex of e_2

Note that $u_0 \neq u_1 \neq u_2$



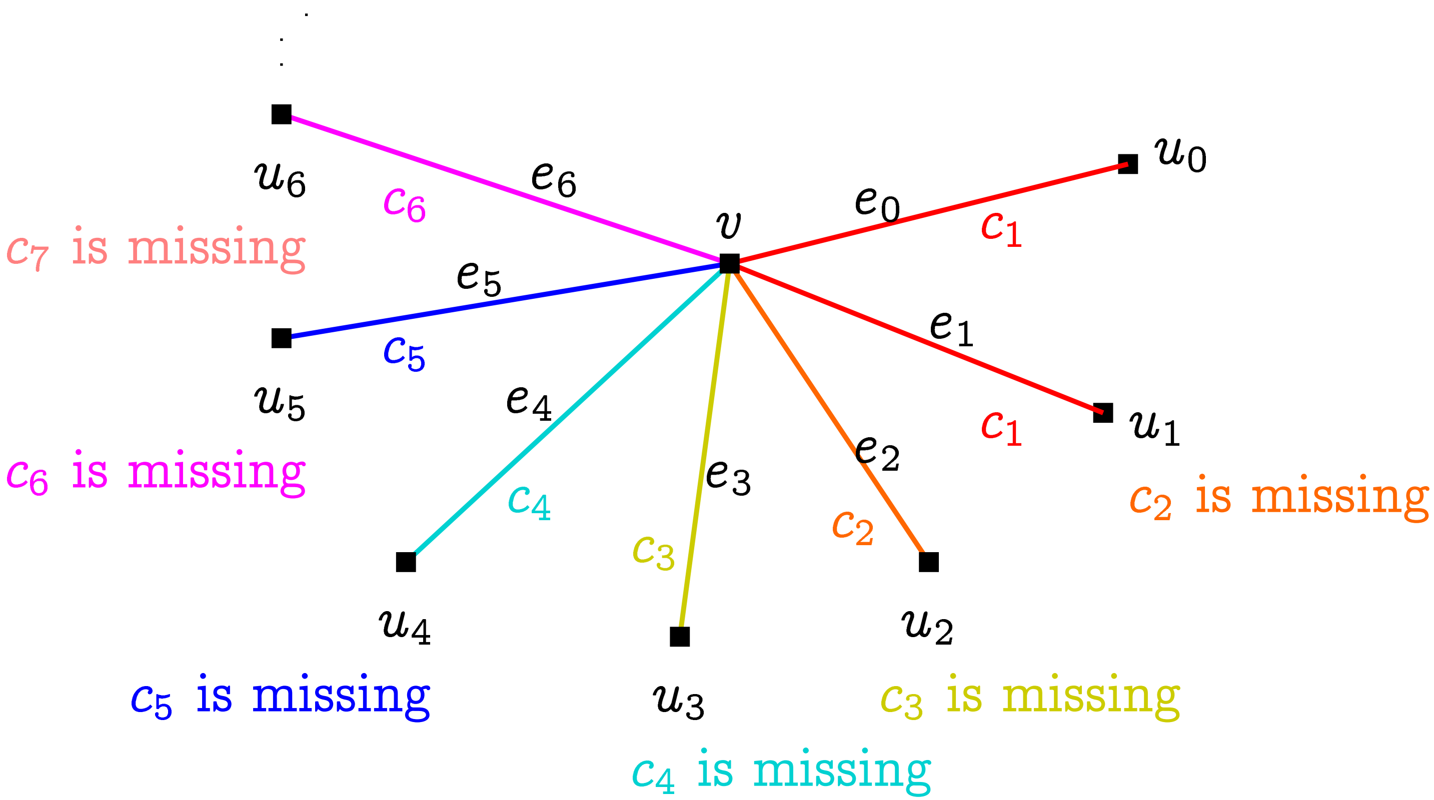
Let color c_3 be missing at u_2



Let color c_3 be missing at u_2

Let e_3 be incident to v, such that $\gamma(e_3) = c_3$

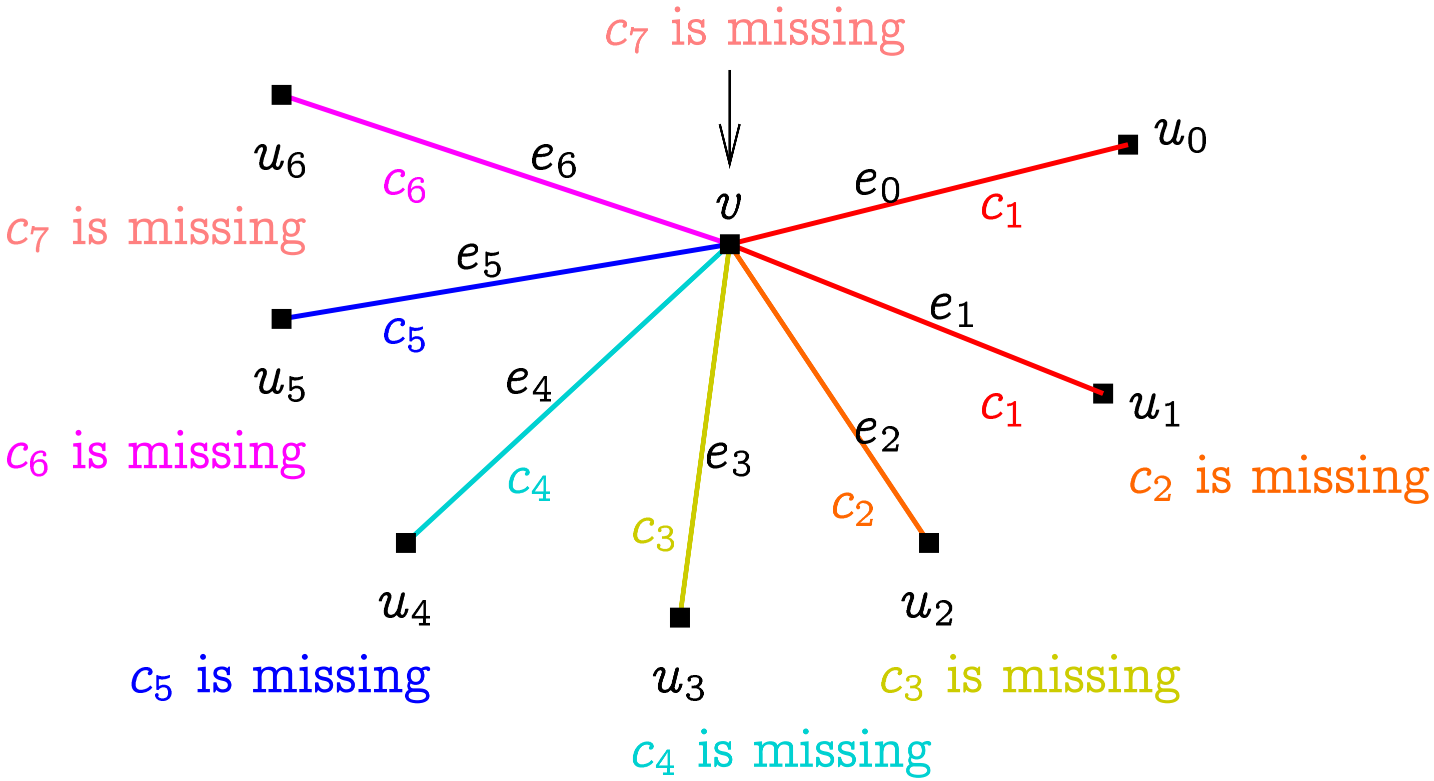
Let u_3 be the other end-vertex of e_3



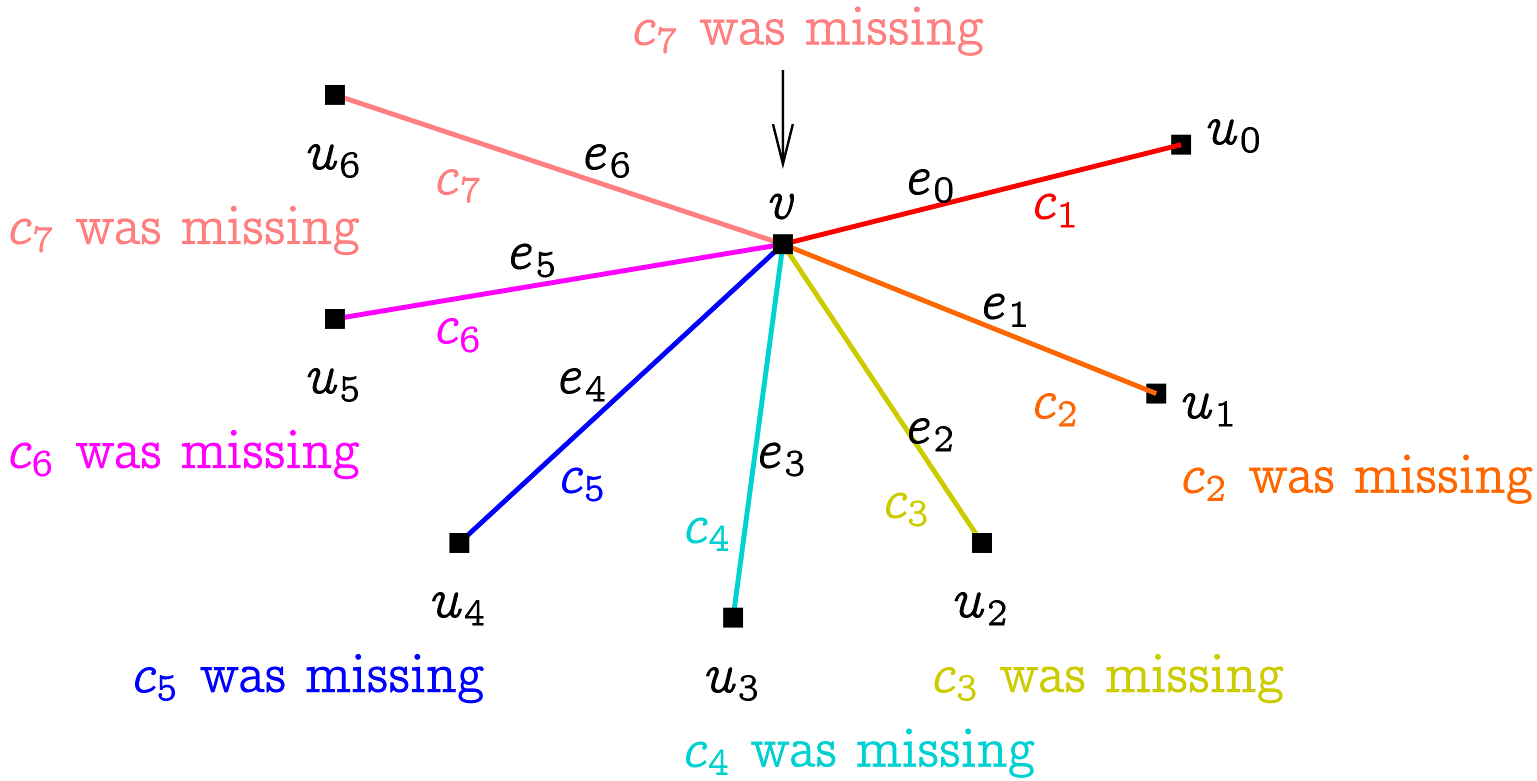
etc (all colors are different)

- $u_0 \neq u_1 \neq u_2 \neq u_3 \neq \dots$
- But $u_i = u_j$ is possible if $j - i \geq 2$.
- We can arrive at the same u up to $\mu(G)$ times.
- Each time, we choose a different color c as the missing one.
- This is possible, because at least $\mu(G)$ colors are missing at each vertex.

- The process of alternately picking colors and edges cannot go on forever.
- It can stop in two ways:
  1. There is no suitable edge: After choosing c_{i+1}, there is no edge e_{i+1} incident with v, that is of color c_{i+1}.
  2. There is no suitable color: all colors missing at u_i have already been picked.
     - In this case, we choose c_{i+1} anyway, but will still stop.



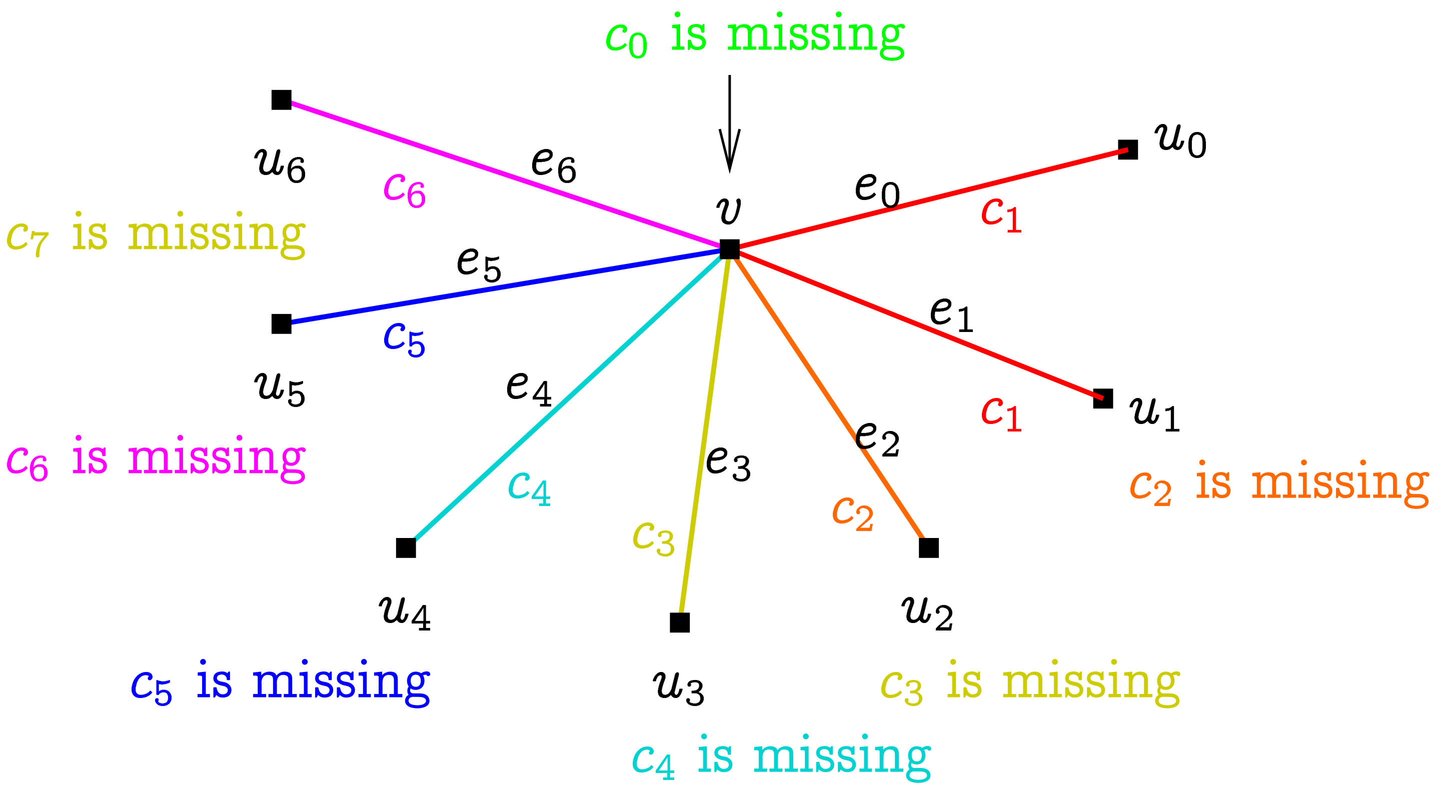
First case (no suitable edge) is simple



Recolor. $\tilde{\gamma}(v)$ increases

$\tilde{\gamma}(\cdot)$ does not decrease for any vertex

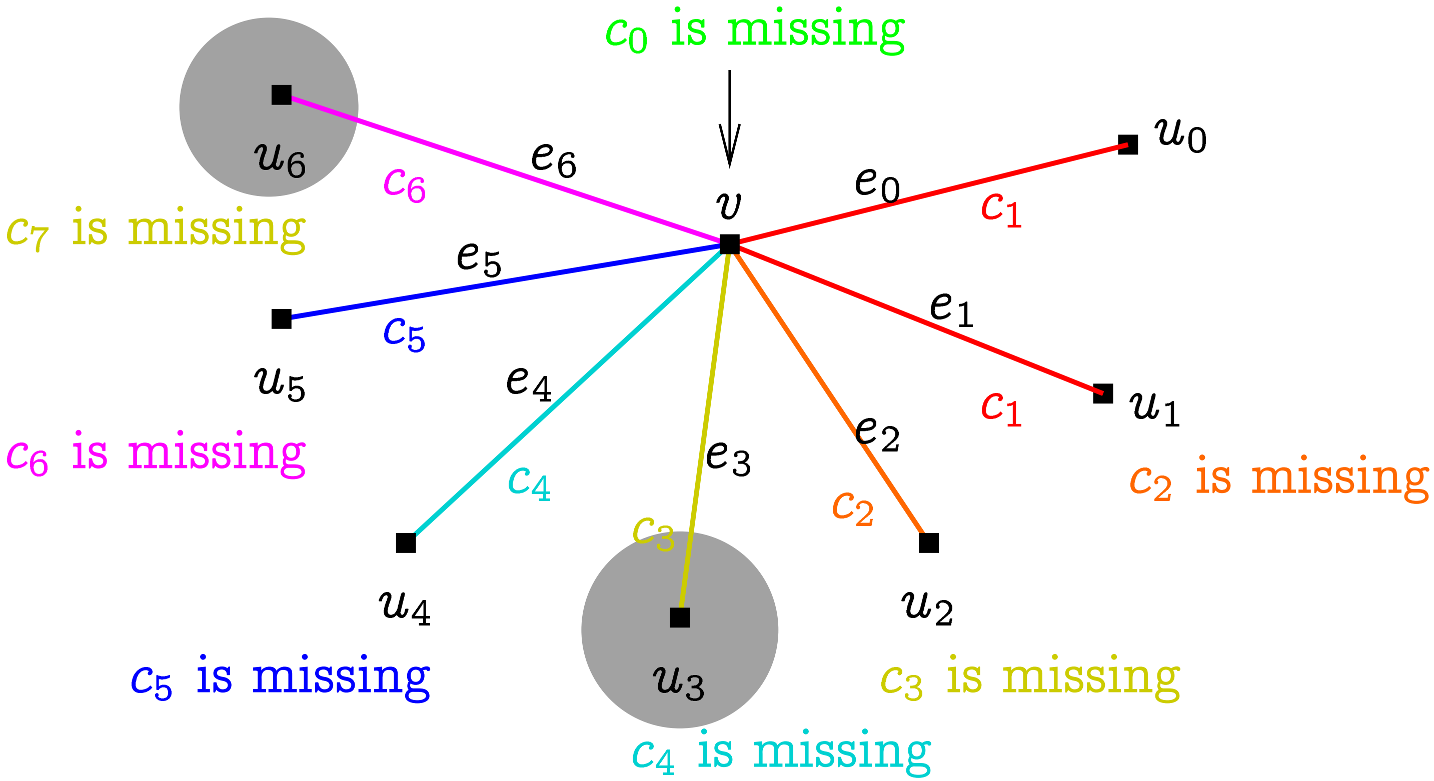
Hence γ was not optimal



Second case. Let c_0 be missing by v

$c_{i+1} = c_k$

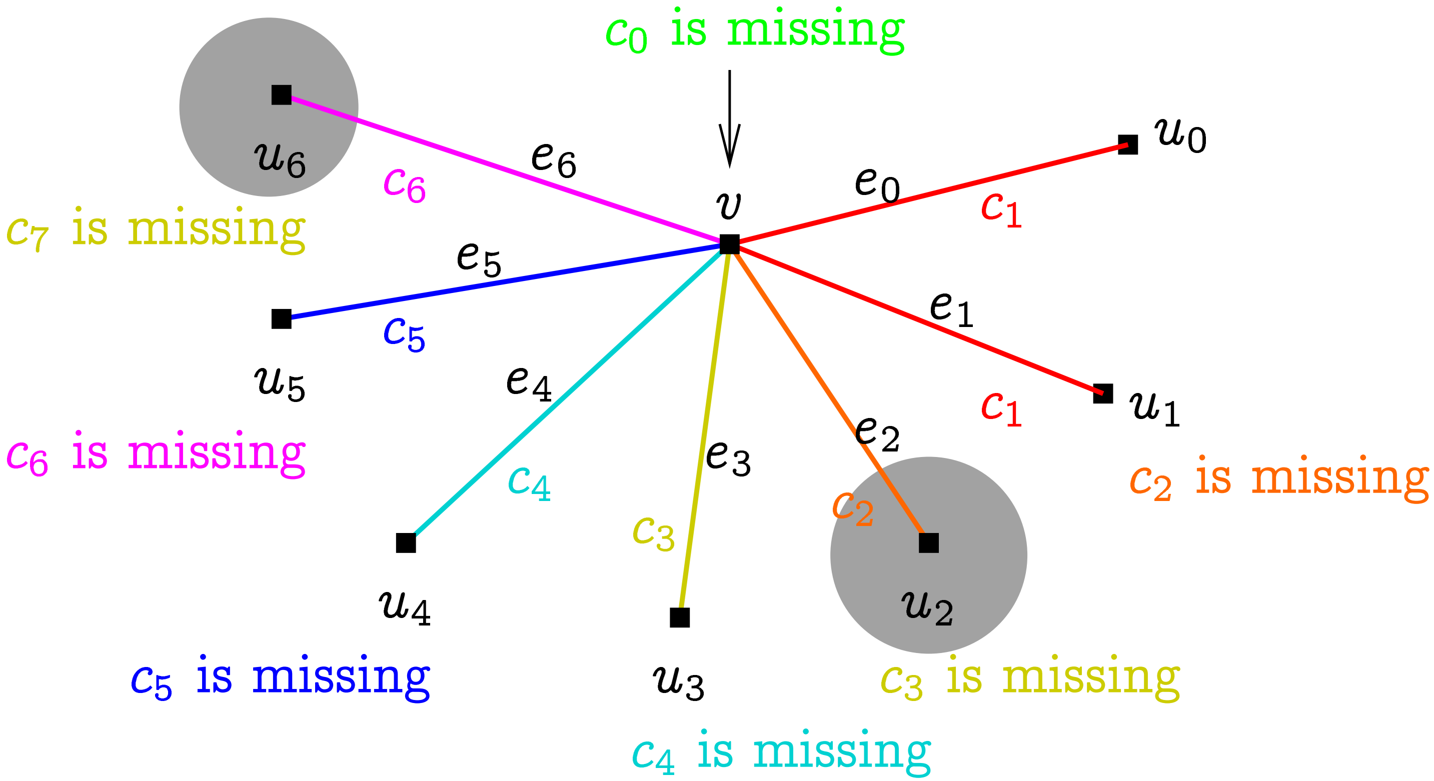
Consider u_i, u_k and u_{k-1}



$u_i \neq u_k$ because

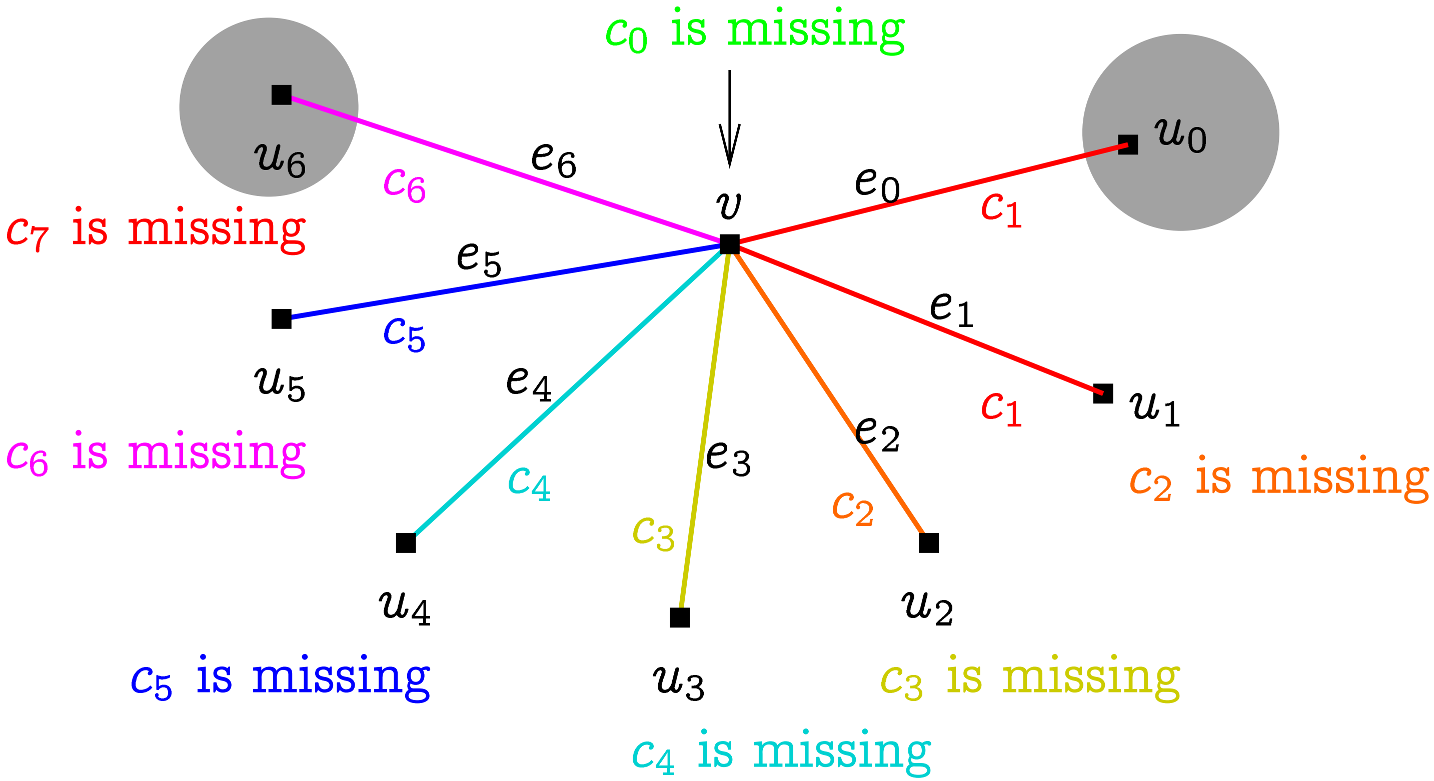
$c_k = c_{i+1}$ is present at u_k

$c_k = c_{i+1}$ is missing at u_i



$u_i \neq u_{k-1}$ because

we will not choose the same color at the same node twice



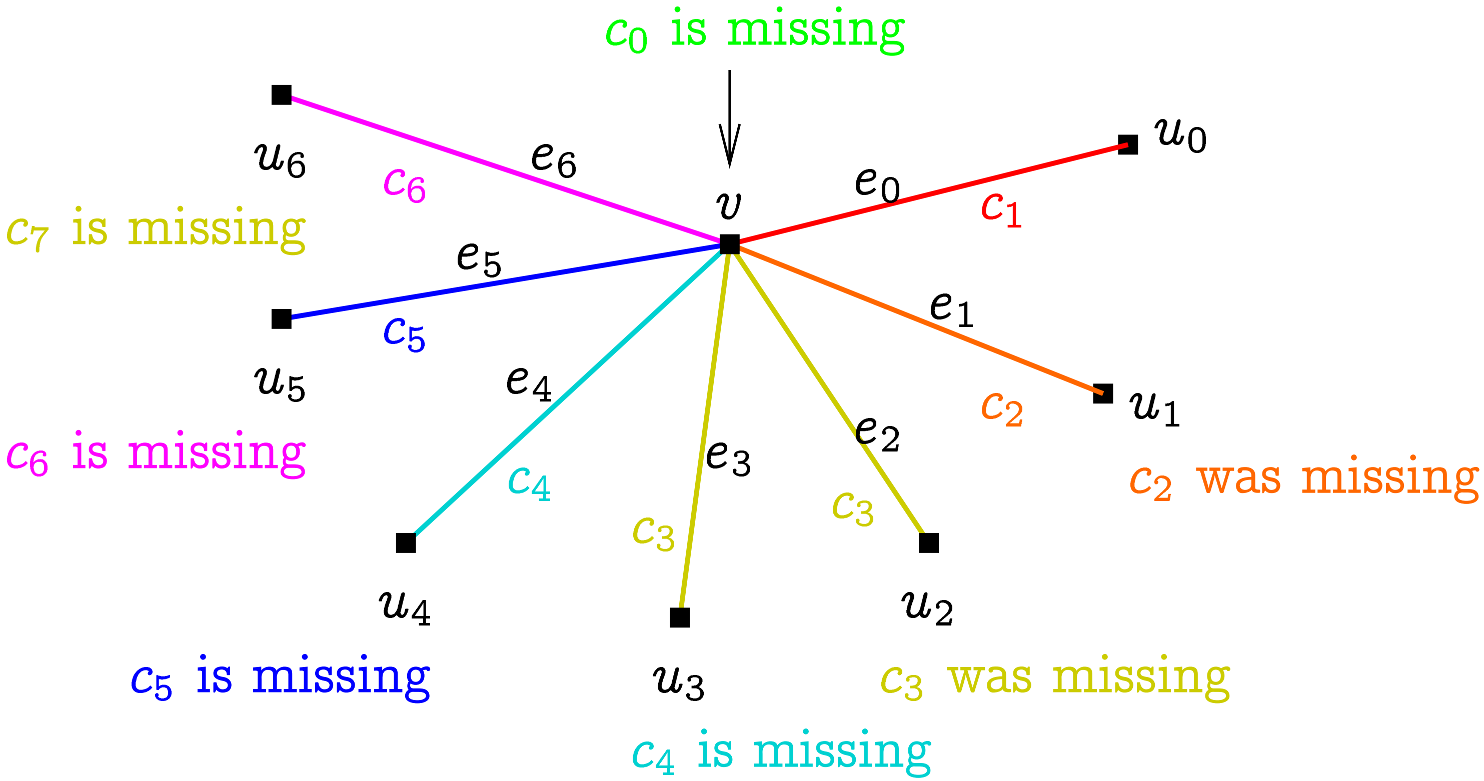
if $k = 1$ then still $u_i \neq u_{k-1}$ because

$c_{i+1} = c_1$ is present at u_0, absent at u_i

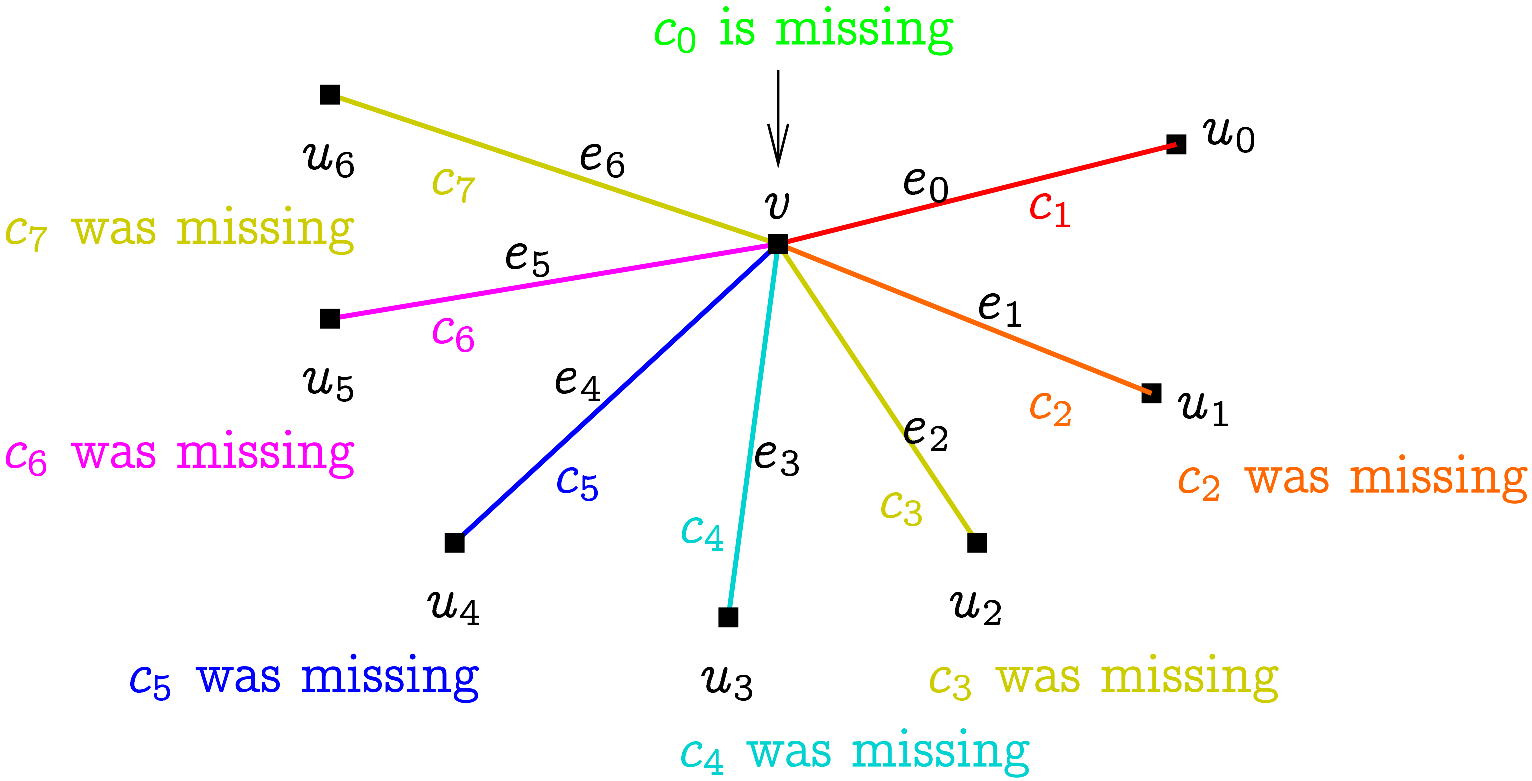
We now consider two colorings obtained from γ as follows:

- For γ', recolor e_j with c_{j+1} for $1 \leq j \leq k-1$.
- For γ'', recolor e_j with c_{j+1} for $1 \leq j \leq i$.

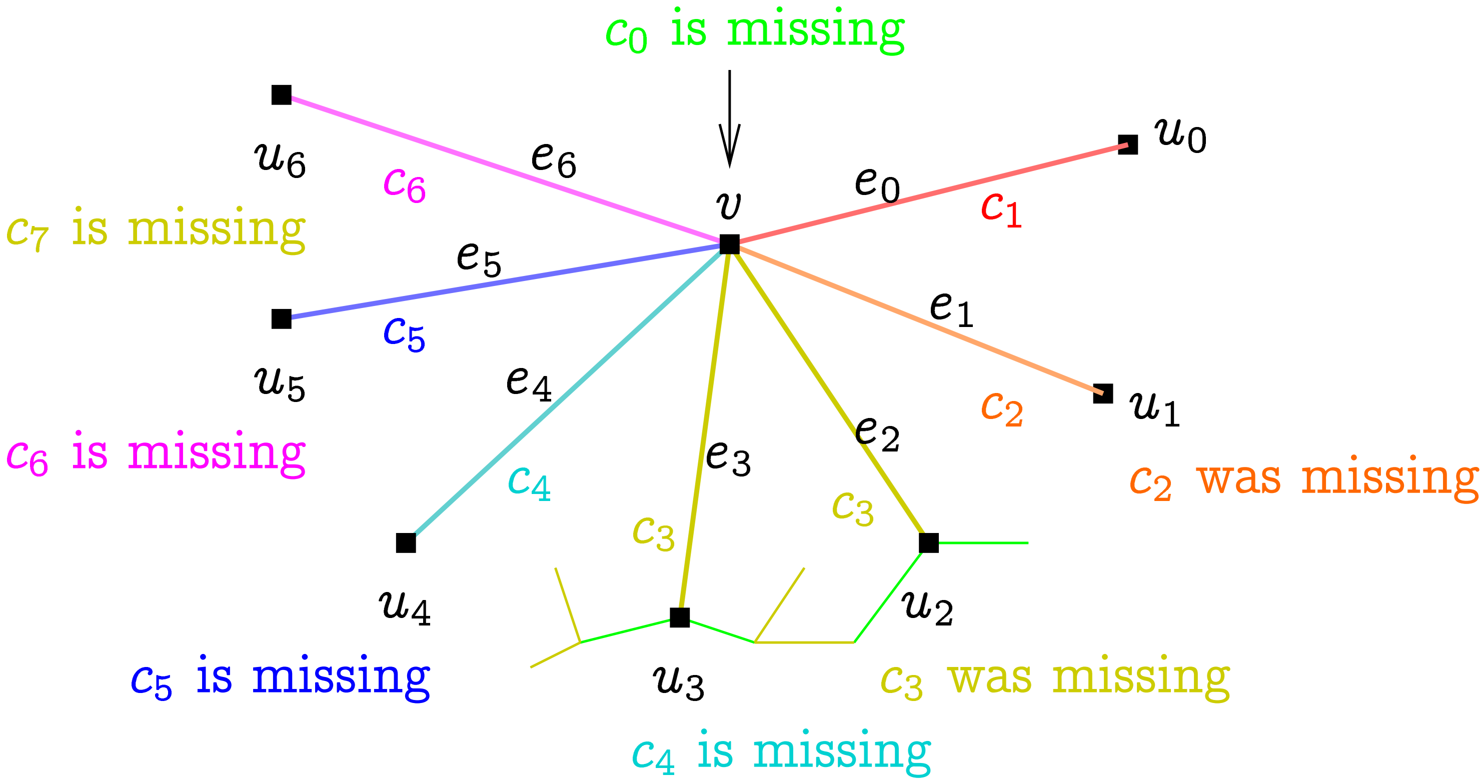
Both of these colorings are optimal, too, because $\tilde{\gamma}(\cdot)$ will not decrease for any vertex.



For γ', recolor e_j with c_{j+1} for $1 \leq j \leq k-1$

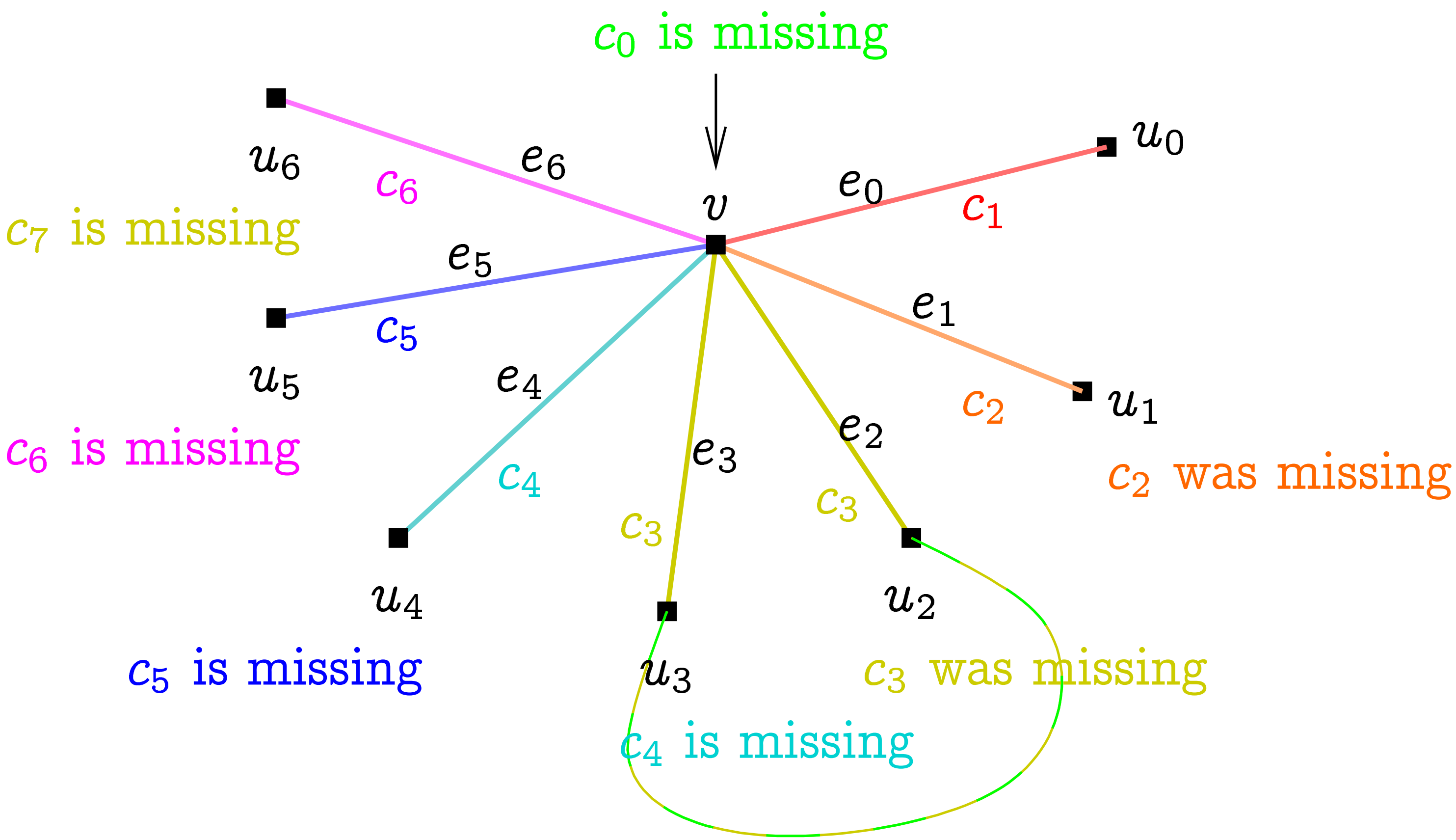


For γ'', recolor e_j with c_{j+1} for $1 \leq j \leq i$



Consider the graph $(V, \gamma'^{-1}(\{c_0, c_k\})$

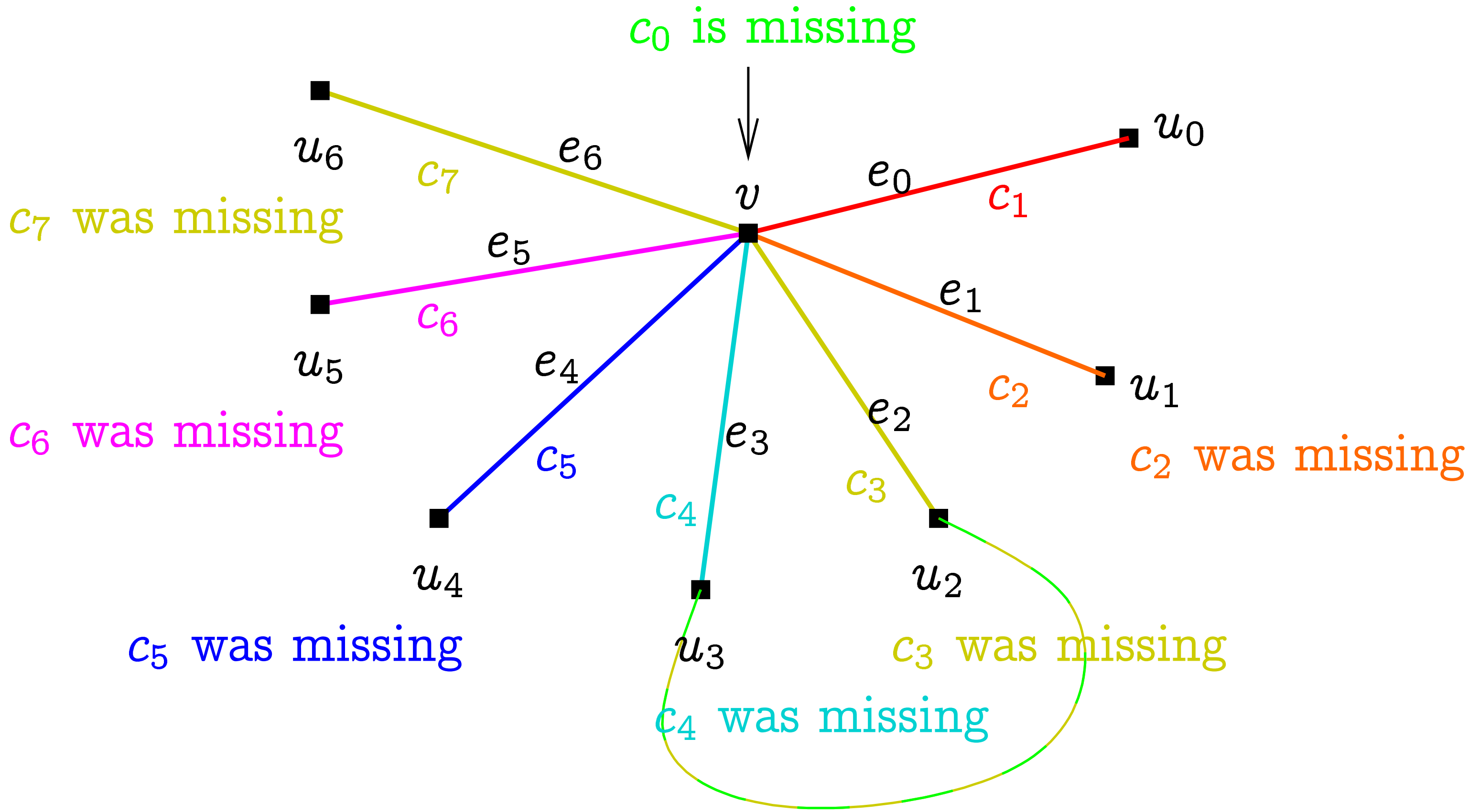
Consider the connected component with v in it



Consider the graph $(V, \gamma'^{-1}(\{c_0, c_k\})$

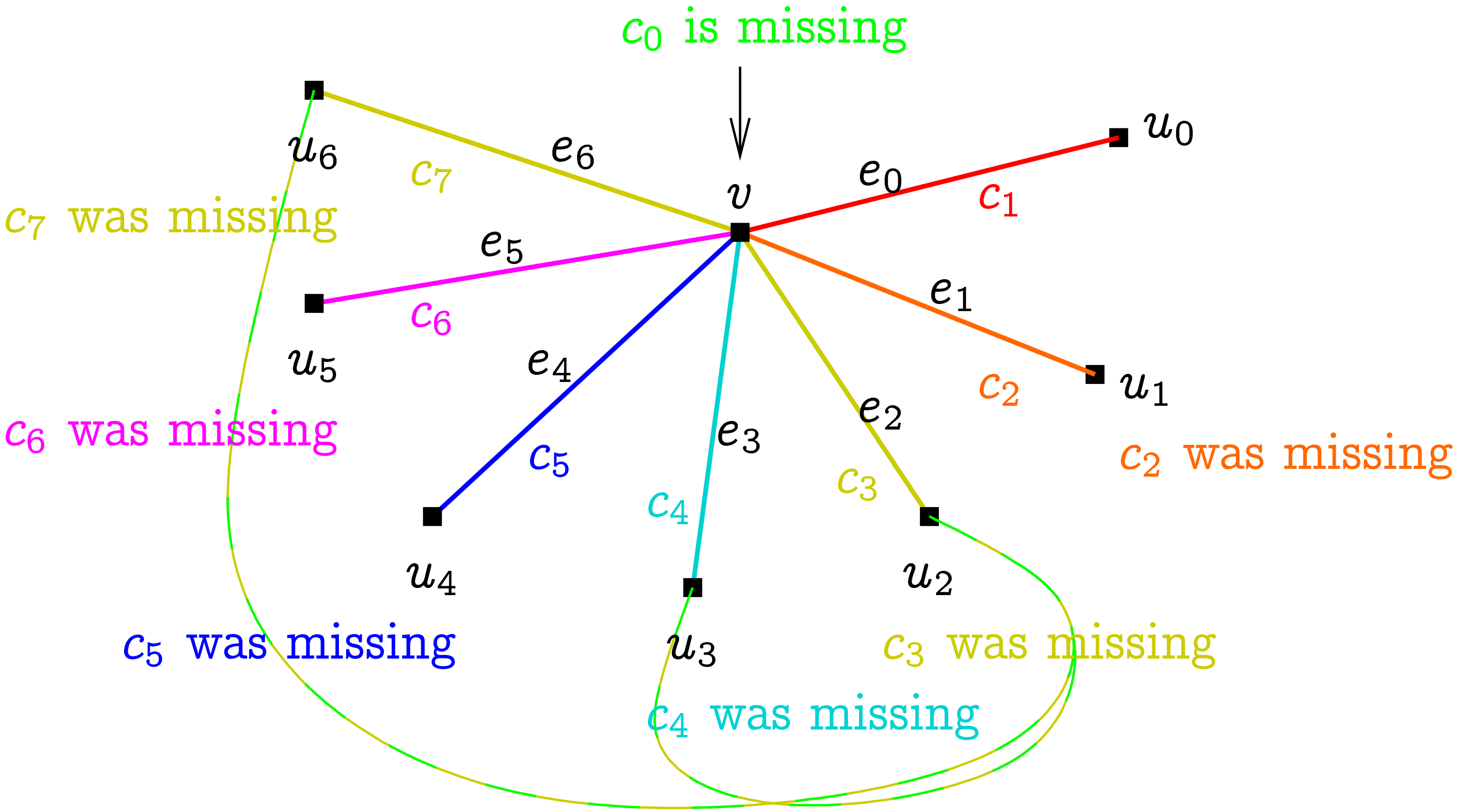
Consider the connected component with v in it

By lemma 2, it is an odd cycle



This path of alternating colors between u_{k-1} and u_k also exists according to γ''

There are no other edges incident to u_k or u_{k-1} that have the color c_0 or c_k



Similarly, consider $\gamma''^{-1}(\{c_0, c_k\})$

The component connecting v must again be a cycle

Not a cycle — degree of u_k is 1. $\square$