Probabilistic proofs

A vertex colouring with k colours of a graph $G = (V, E)$ is a mapping $\gamma : V \to \{1, \ldots, k\}$, such that $\gamma(u) \neq \gamma(v)$ for any edge $(u, v) \in E$.

The chromatic number $\chi(G)$ of a graph G is the smallest k, such that G has vertex colouring with k colours.

The girth $g(G)$ of a graph G is the length of the shortest cycle in G.

A graph with a large girth "locally looks" like a tree. Trees can be coloured with two colours. Nevertheless

Theorem. For any $k \in \mathbb{N}$ there exists a graph G, such that $g(G) > k$ and $\chi(G) > k$.

Proof follows...

A *probability distribution* on a set $\mathbf{X}$ is a function $\mu : \mathbf{X} \longrightarrow [0,1]$, such that $\sum_{x \in \mathbf{X}} \mu(x) = 1$.

(we assume that X is finite)

An *event* on a set $\mathbf{X}$ is a subset $A \subseteq \mathbf{X}$.

Let μ be fixed. Then $\mathbf{P}(A) = \sum_{x \in A} \mu(x)$.

If $A, B \subseteq \mathbf{X}$, then $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$.

Let $F : \mathbf{X} \longrightarrow \mathbb{R}^+$. F can be seen as a *random variable* with the distribution μ.

The *mean* of F is $\mathbf{E}(F) = \sum_{x \in \mathbf{X}} \mu(x) F(x)$.

$\mathbf{E}$ is linear: $\mathbf{E}(F + F') = \mathbf{E}(F) + \mathbf{E}(F')$. This holds even if F and F' are not independent.

If $F(\mathbf{X}) \subseteq \{0, 1\}$, then $\mathbf{E}(F) = \mathbf{P}(F = 1)$.

If $A \subseteq \mathbf{X}$, then let χ_A be its characteristic function. Then $\mathbf{E}(\chi_A) = \mathbf{P}(A)$.

If $F(\mathbf{X}) \subseteq \mathbb{N}$, then $\mathbf{E}(F) \geq \mathbf{P}(F > 0)$.

Lemma (Markov's inequality). Let F be a random variable and $a > 0$. Then

$$\mathbf{P}(F \geq a) \leq \mathbf{E}(F)/a \ .$$

Proof.

$$\mathbf{E}(F) = \sum_{x \in \mathbf{X}} \mu(x) F(x) \geq \sum_{\substack{x \in \mathbf{X} \\ F(x) \geq a}} \mu(x) F(x)$$

$$\geq \sum_{\substack{x \in \mathbf{X} \\ F(x) \geq a}} \mu(x) \cdot a = \mathbf{P}(F \geq a) \cdot a \ . \qquad \square$$

This inequality is helpful for showing that $\mathbf{P}(F < a)$ is large.

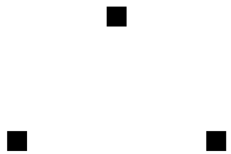
Let $p \in [0,1]$. Define the following probability distribution $\mathcal{G}(n,p)$ on the set $\mathbf{G}_n$ of n-vertex labeled graphs:

Picking G according to $\mathcal{G}(n,p)$ (denote $G \leftarrow \mathcal{G}(n,p)$) proceeds as follows:

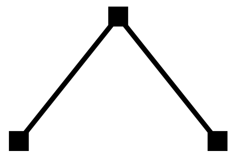
- $V(G) := \{v_1, \ldots, v_n\}$. Let $E(G) := \emptyset$.
- For all $i \in \{1, \ldots, n-1\}$ and $j \in \{i+1, \ldots, n\}$:
  - Toss a coin, where the probability of *heads* is p.
  - If the result was *heads*, then $E(G) := E(G) \cup \{(v_i, v_j)\}$.
  - The coin-tosses must be mutually independent.

In the following denote $q = 1 - p$.

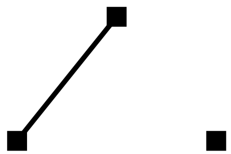
Example. Picking an (unlabeled) graph according to $\mathcal{G}(3, p)$ gives us the following graphs with the following probabilities:



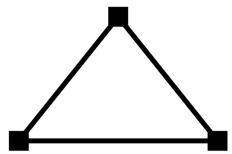
q^3



$3p^2q$



$3pq^2$



p^3

$\mathbf{E}(\Delta) = 3pq^2 + 6p^2q + p^3$. If $p = q = 1/2$, then $\mathbf{E}(\Delta) = 5/4$.

Let $G \leftarrow \mathcal{G}(n, p)$. Let H be a fixed graph with $n' \leq n$ vertices and m' edges.

Let $\psi : V(H) \longrightarrow V(G)$ be an injective function. The probability that ψ locates a copy of H as a subgraph of G, is $p^{m'}$.

The probability that ψ locates an induced subgraph H of G is $p^{m'} q^{\binom{n'}{2} - m'}$.

In general, $\mathbf{P}(H \hookrightarrow G) \leq \sum_{\substack{U \subseteq V(G) \\ |U| = n'}} \mathbf{P}(H \cong G[U])$.

This sum is the average number of times H occurs in G as an induced subgraph.

Lemma. Let $G \leftarrow \mathcal{G}(n,p)$. The average number of k-vertex cliques in G is $\binom{n}{k}p^{\binom{k}{2}}$ and the average number of k-vertex independent sets is $\binom{n}{k}q^{\binom{k}{2}}$.

Proof. Fix $U \subseteq V(G)$, such that $|U| = k$. The probability that U is a clique is $p^{\binom{k}{2}}$.

The *average number of cliques in position* U is $p^{\binom{k}{2}}$. There are $\binom{n}{k}$ possible positions, and we can just add the averages. $\square$

Let $\alpha(G)$ be the size of the largest independent set that G contains. Then $\mathbf{P}(\alpha \geq k) \leq \binom{n}{k}q^{\binom{k}{2}}$.

Recall that $\chi(G) \geq n/\alpha(G)$, where n is the number of vertices of G.

Denote

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1) \ .$$

Lemma. Let $G \leftarrow \mathcal{G}(n,p)$. The average number of cycles of length $k \geq 3$ in G is $p^k(n)_k/2k$.

Proof. A cycle of length k is determined by a sequence $(v_1, v_2, \ldots, v_k)$ of different vertices of G.

Such a sequence can be chosen in $(n)_k$ different ways. Each cycle corresponds to $2k$ such sequences.

The probability that G contains the edges (v_1, v_2), (v_2, v_3),..., (v_{k-1}, v_k), (v_k, v_1) is p^k. $\square$

Let $X_k(G)$ be the number of cycles of length at most k in the graph G. If $G \leftarrow \mathcal{G}(n,p)$, then

$$\mathbf{E}[X_k] = \sum_{i=3}^{k} \frac{(n)_i}{2i} p^i \leq \frac{1}{2} \sum_{i=3}^{k} n^i p^i \leq \begin{cases} \frac{k-2}{2} n^k p^k, & \text{if } np \geq 1 \\ \frac{k-2}{2n^3p^3} \cdot \frac{1}{1-np}, & \text{if } np < 1 \end{cases}$$

This is an upper bound for $\mathbf{P}(g \leq k)$.

To show the existence of a graph G with $g(G) \geq k$ and $\chi(G) \geq k$ we could try to fix n and p so, that

$$\mathbf{P}(g \leq k-1) + \mathbf{P}(\alpha \geq n/k) < 1 \ .$$

It turns out that there are no such n and p...

We will show that we can fix n and p so, that

- $\mathbf{P}(X_k \geq n/2) < 1/2$;
- $\mathbf{P}(\alpha \geq n/2k) < 1/2$.

  We fix p as a function of n so, that both of those probabilities approach 0 if $n \to \infty$.

Hence there exists an n-vertex graph G containing less than $n/2$ cycles of length $\geq k$, and no independent set of size $n/2k$. Let H be a graph obtained from G by removing one vertex from each of those short cycles.

$|V(H)| > n/2$. Obviously $g(H) > k$ and $\alpha(H) < n/2k < |V(H)|/k$. Hence k colours are not sufficient to colour H.

Fix $\varepsilon \in \mathbb{R}$, such that $0 < \varepsilon < 1/k$. Let $p = n^{\varepsilon-1}$. Then $0 < p \leq 1$.

$$\mathbf{P}(X_k \geq n/2) \leq \mathbf{E}[X_k]/(n/2) \leq \frac{k-2}{2\cdot(n/2)} n^k p^k = (k-2)(np)^k/n = (k-2)n^{k\varepsilon-1}$$

- because $np = n^\varepsilon \geq n^0 = 1$.

As $k\varepsilon - 1 < 0$, the above expression tends to 0 if $n \to \infty$.

Let r be such, that $n \geq r \geq n/2k$.

Note that $p \geq (6k \ln n)/n$ if n is large enough.

$$\mathbf{P}(\alpha \geq r) \leq \binom{n}{r} q^{\binom{r}{2}} \leq n^r q^{\frac{r(r-1)}{2}} = (nq^{(r-1)/2})^r \leq$$
$$(ne^{-p(r-1)/2})^r = (ne^{-pr/2+p/2})^r \leq (ne^{-(3/2)\ln n+p/2})^r \leq$$
$$(nn^{-3/2}e^{1/2})^r = (e/n)^{r/2} \quad .$$

- because $1 - p \leq e^{-p}$ if $0 \leq p \leq 1$
- because of the lower bounds on r and p

If $n \to \infty$, then $e/n \to 0$ and $r/2 \to \infty$. Hence the whole expression tends to 0. □

Let us now consider simple graphs with countably many vertices. In particular, consider graphs distributed according to $\mathcal{G}(\mathbb{N}, 1/2)$.

Theorem. Let $G_1 \leftarrow \mathcal{G}(\mathbb{N}, 1/2)$ and $G_2 \leftarrow \mathcal{G}(\mathbb{N}, 1/2)$, where G_1 and G_2 are two independent random variables. Then the following event occurs with probability 1:

There exists an isomorphism from G_1 to G_2.

In other words, there exists exactly one random countably infinite simple graph.

Consider the following property (*), that a graph $G = (V, E)$ may or may not satisfy:

- for any finite $U, W \subseteq V$, where $U \cap W = \emptyset$
- exists $z \in V \backslash (U \cup W)$
- such that
  - for all $u \in U$, $(u, z) \in V$;
  - for all $w \in W$, $(w, z) \notin V$.

Lemma. Let $G \leftarrow \mathcal{G}(\mathbb{N}, 1/2)$. Then G satisfies (*) with probability 1.

Proof. Fix U and W. If we also fix z, then the probability of (*) holding is $1/2^{|U|+|W|}$. We have infinitely many choices for z, thus the probability of (*) holding for some choice of z is 1. $\square$

Lemma. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two countably infinite simple graphs that satisfy (*). Then $G_1 \cong G_2$.

Proof. Identify both V_1 and V_2 with $\mathbb{N}$.

We construct the isomorphism $\varphi : V_1 \to V_2$ in rounds.

- In the beginning, φ is everywhere undefined. Each round defines φ for one element of V_1 (and V_2).
- For any $v_1 \in V_1$, $\varphi(v_1)$ will be defined after a finite number of rounds.
- For any $v_2 \in V_2$, $\varphi^{-1}(v_2)$ will be defined after a finite number of rounds.

After countably many rounds, we have a uniquely defined bijection between V_1 and V_2. It will be an isomorphism.

n-th round (for odd n):

- Let $x_n = \min\{x \in V_1 \mid \varphi(x) \text{ is undefined}\}$.
- Let $U_n = \{v \in V_1 \mid (x_n, v) \in E_1 \wedge \varphi(v) \text{ is defined}\}$.
- Let $W_n = \{v \in V_1 \mid (x_n, v) \notin E_1 \wedge \varphi(v) \text{ is defined}\}$.
- By (*) for G_2, there exists some $y_n \in V_2 \backslash (\varphi(U_n) \cup \varphi(W_n))$, such that y_n is connected to all vertices in $\varphi(U_n)$ and to no vertices in $\varphi(W_n)$.
  - φ^{-1} is defined only for vertices in $\varphi(U_n) \cup \varphi(W_n)$,
  - hence $\varphi^{-1}(y_n)$ is not defined.
- Let the new value of φ be $\varphi[x_n \mapsto y_n]$.

n-th round (for even n) (just swap G_1 and G_2):

- Let $y_n = \min\{y \in V_2 \,|\, \varphi^{-1}(y) \text{ is undefined}\}$.
- Let $U_n = \{v \in V_2 \,|\, (y_n, v) \in E_2 \wedge \varphi^{-1}(v) \text{ is defined}\}$.
- Let $W_n = \{v \in V_2 \,|\, (y_n, v) \notin E_2 \wedge \varphi(v) \text{ is defined}\}$.
- By (*) for G_1, there exists some $x_n \in V_1 \backslash (\varphi^{-1}(U_n) \cup \varphi^{-1}(W_n))$, such that x_n is connected to all vertices in $\varphi^{-1}(U_n)$ and to no vertices in $\varphi^{-1}(W_n)$.
  - φ is defined only for vertices in $\varphi^{-1}(U_n) \cup \varphi^{-1}(W_n)$,
  - hence $\varphi(x_n)$ is not defined.
- Let the new value of φ be $\varphi[x_n \mapsto y_n]$. $\square$

From those two lemmas, the theorem immediately follows.

$\square$