Coloring vertices

(Correct) vertex coloring of the vertices of graph $G = (V, E)$ using k colors is a function $c : V \longrightarrow \{1, \ldots, k\}$ such that

- for each $e \in E$, where $\mathcal{E}(e) = \{u, v\}$, we have $c(u) \neq c(v)$.

I.e. end vertices of all the edges are colored with different colors.

Graph G is *colorable with k colors* if there exists a correct vertex coloring of G using k colors.

(Vertex) chromatic number of graph G is the smallest number k such that the graph is colorable with k colors. This number is denoted $\chi(G)$.

Graphs colorable with k colors may be called *k-partite*.

Let $\Delta(G)$ denote the maximal vertex degree of G.

Theorem. Graph $G = (V, E)$ without loops is colorable with $\Delta(G) + 1$ colors.

Proof. Induction over $|V|$.

Base. $|V| = 1$. Obvious.

Step. $|V| > 1$. Let $v \in V$. Induction hypothesis implies that $G\backslash v$ can be colored with $\Delta(G\backslash v) + 1$ colors. It is also colorable with $\Delta(G) + 1$ colors, since $\Delta(G) \geq \Delta(G\backslash v)$.

Let c be a coloring of $G\backslash v$ with $\Delta(G) + 1$ colors. There exists a color i, such that none of the neighbours of v is of this color. Thus we can use the color i for v. $\square$

Let $G = (V, E)$ be a graph. Vertex subset $S \subseteq V$ is called a *clique*, if any two (different) vertices $u, v \in S$ are joined by an edge in G.

Stating it otherwise, S is a clique if the induced subraph $G[S]$ is a complete one.

Vertex subset $S \subseteq V$ is called an *independet set* if no two (different) vertices $u, v \in S$ are joined by an edge in G.

Stating it otherwise, S is an independent set if the induced subraph $G[S]$ is a null graph.

Recall that the null graph with n vertices is denoted O_n.

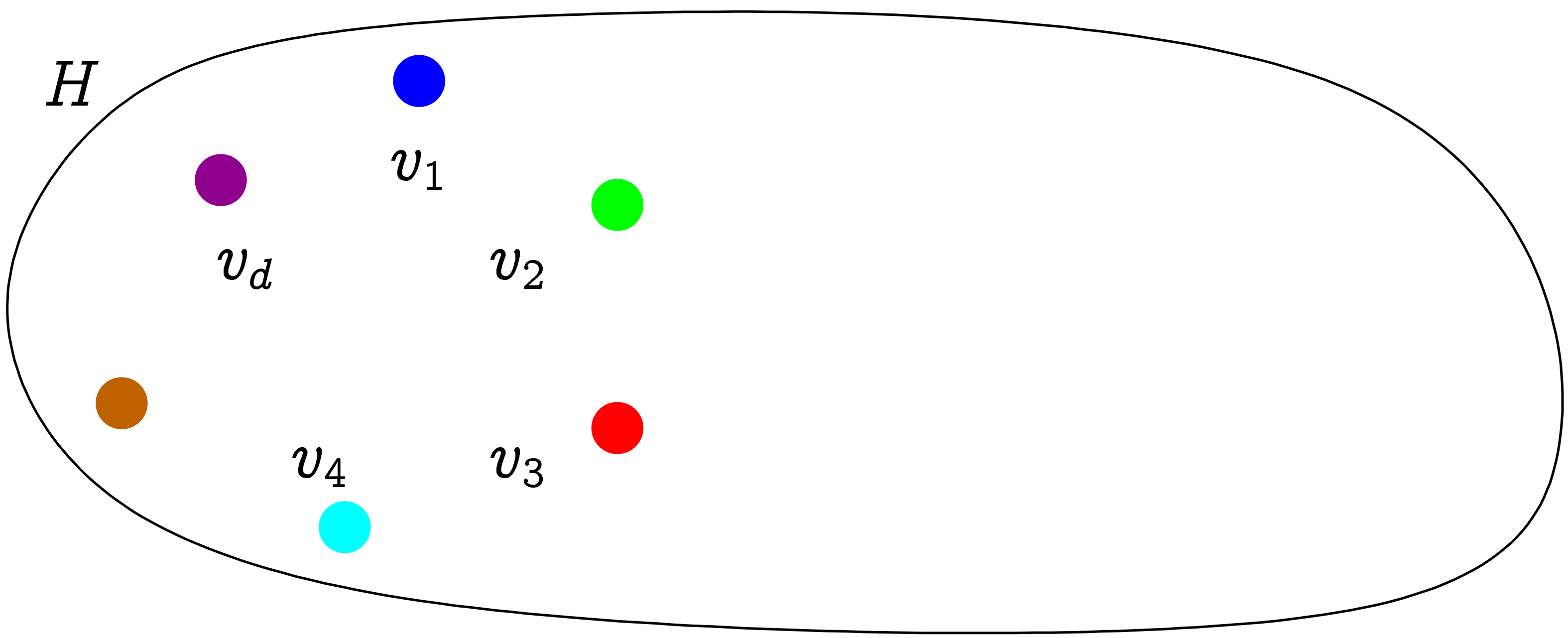
Theorem (Brooks). Let $G = (V, E)$ be a graph without loops. Let $\Delta(G) \leq d$ ($d \geq 3$). If G has no $(d+1)$-element clique, then $\chi(G) \leq d$.

Proof. Assume that the statement of the theorem is wrong and let $G = (V, E)$ be the counterexample with minimal number of vertices. We have

- $\Delta(G) = d \geq 3$;
- no connected component of G is K_{d+1};
- $\chi(G) > \Delta(G)$.

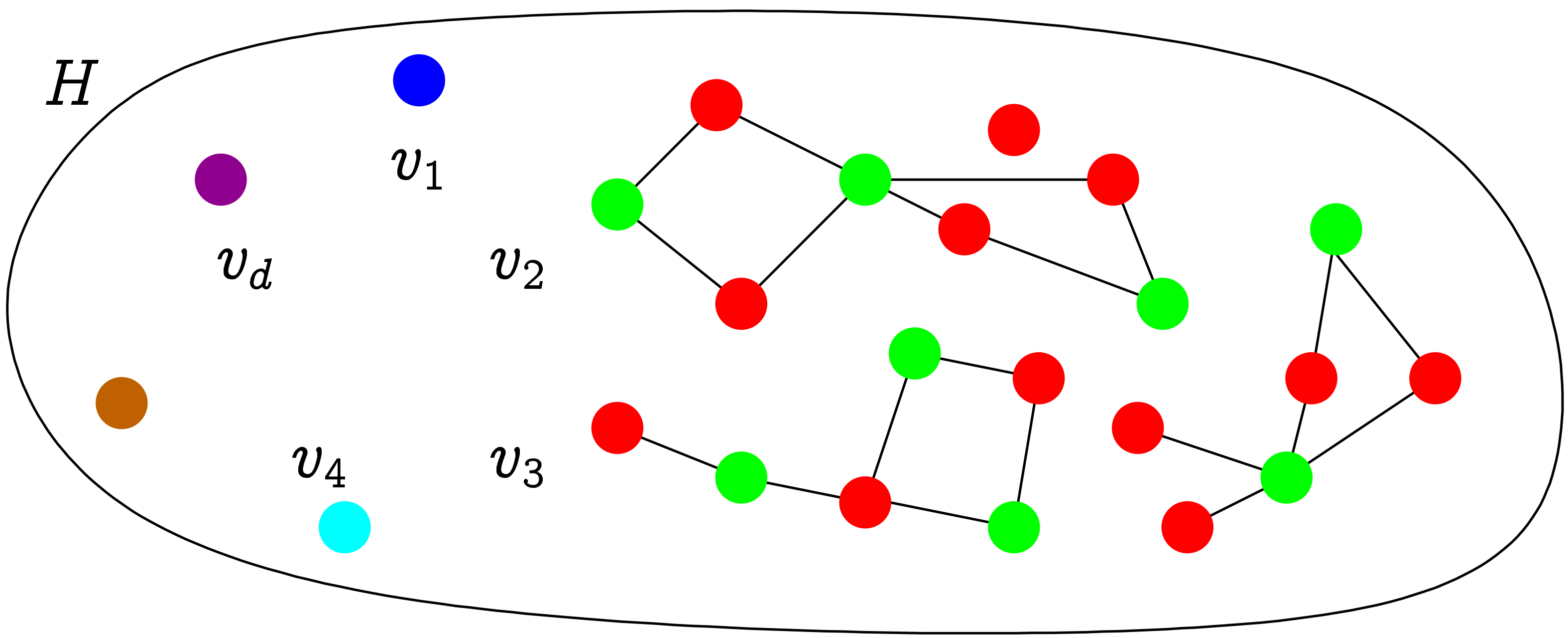
Let $v \in V$ be a vertex. Let $H = G \backslash v$ and let c be some coloring of H using d colors.

v has d neighbours and they all are colored differently. Let $N(v) = \{v_1, \ldots, v_d\}$, such that $c(v_i) = i$.

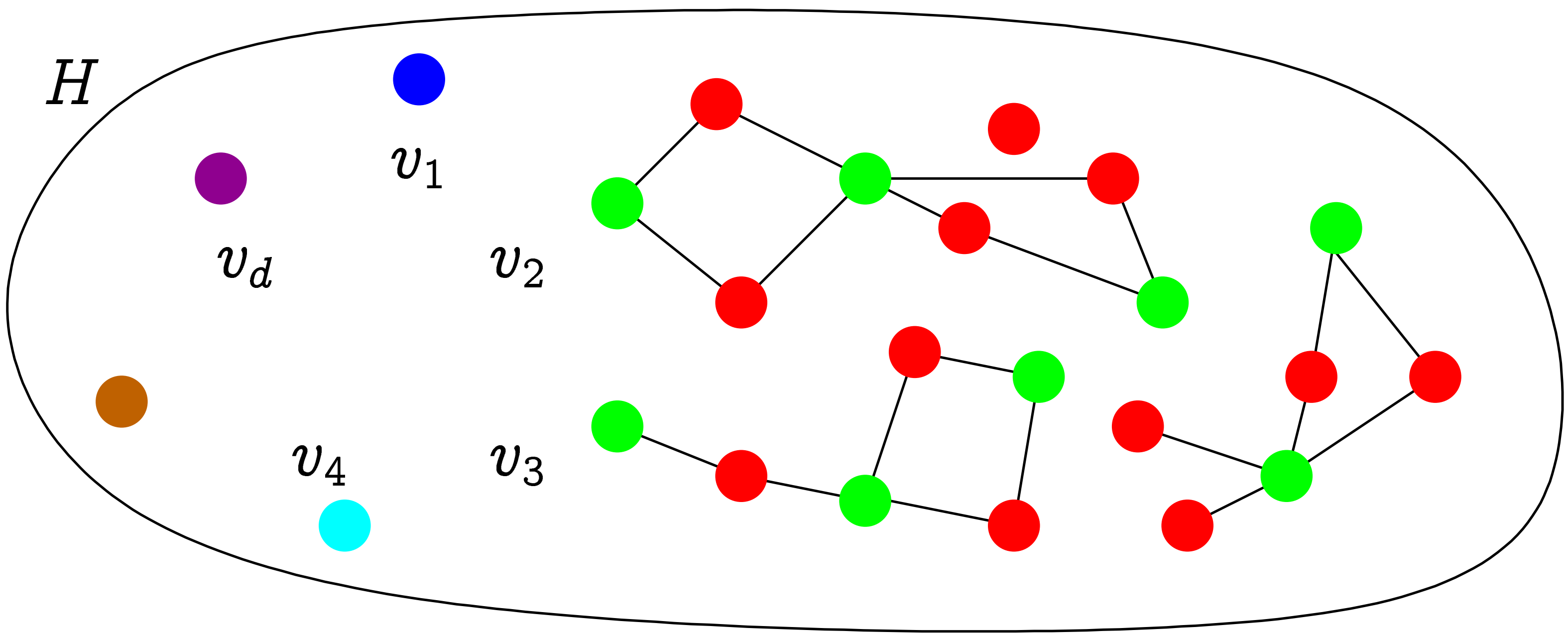


To prove the theorem, we must show that H has a coloring c' using d colors such that two neighbours of v are colored the same.

Let i and j be two colors and let H_{ij} be the subgraph of H induced by the vertices of colors i and j.

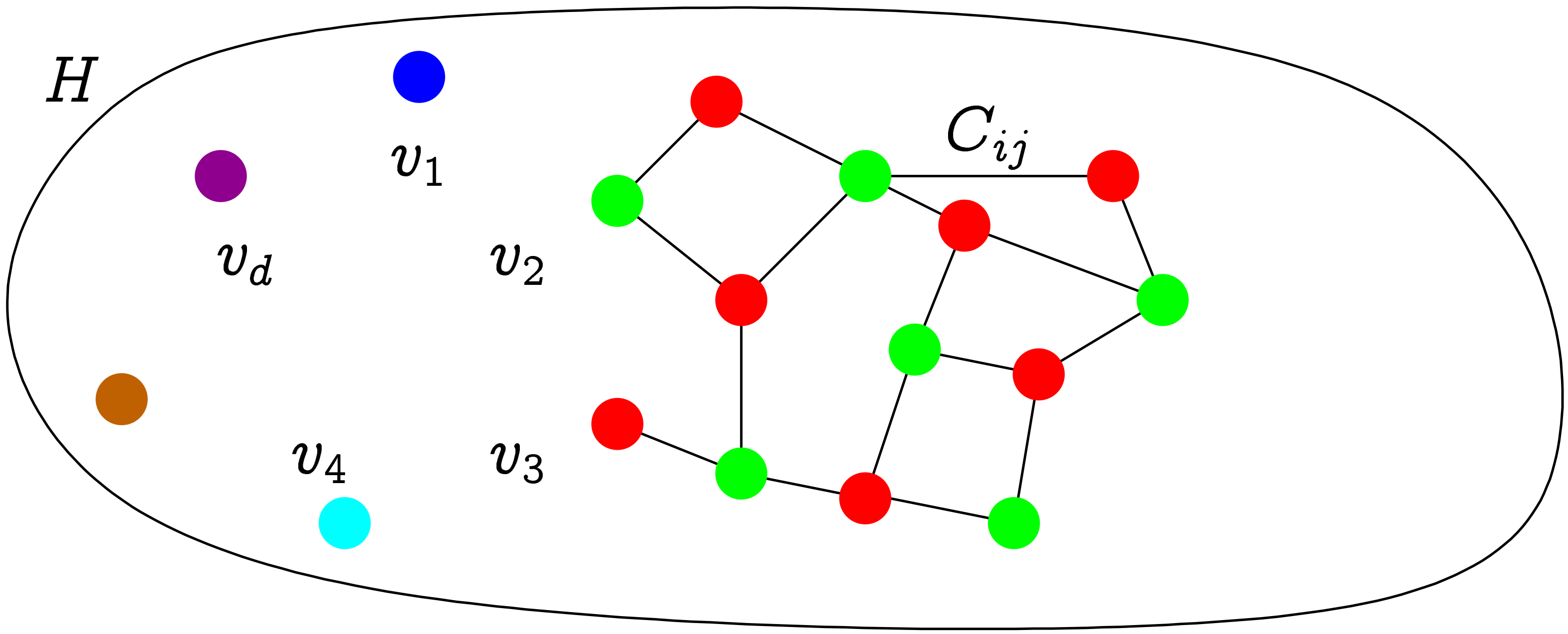


If we exchange the colors of vertices in some connected compon ent of H_{ij}, we get a correct coloring.



Thus v_i and v_j must be located in the same connected component of H_{ij}.

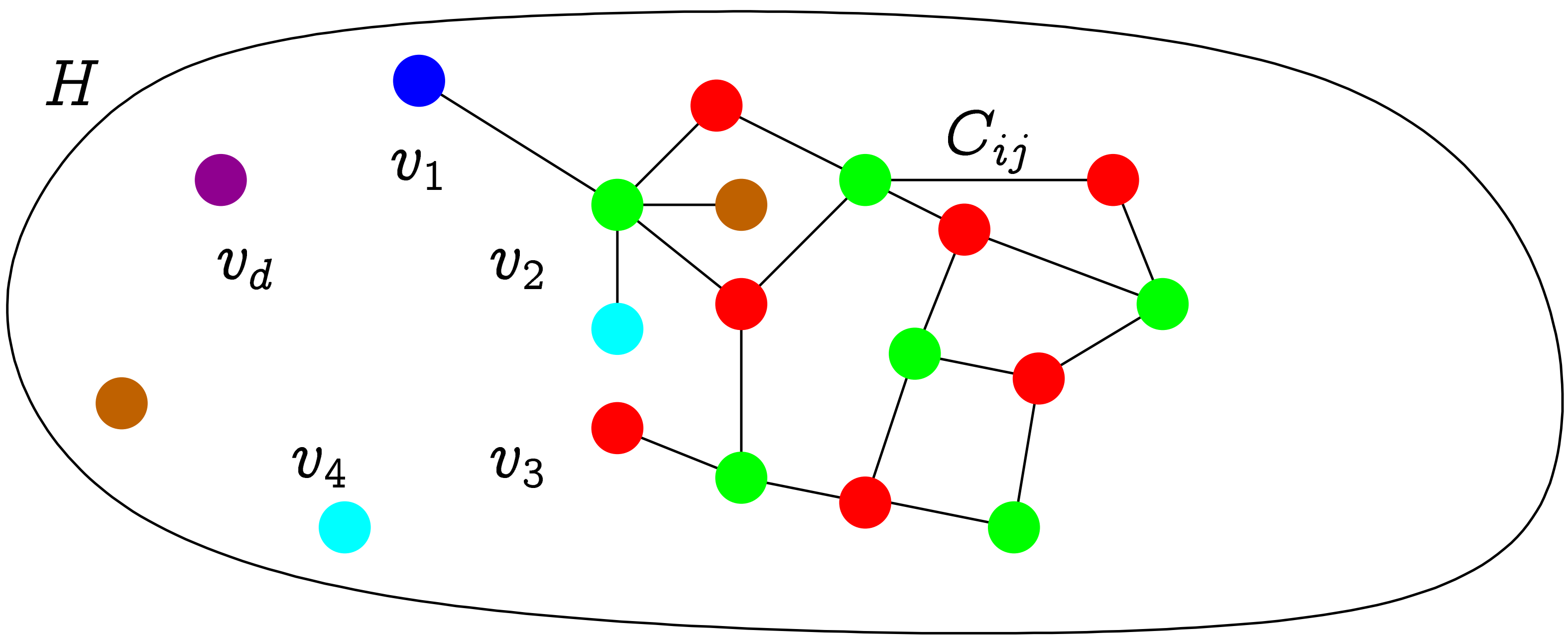
Denote this component by C_{ij}.



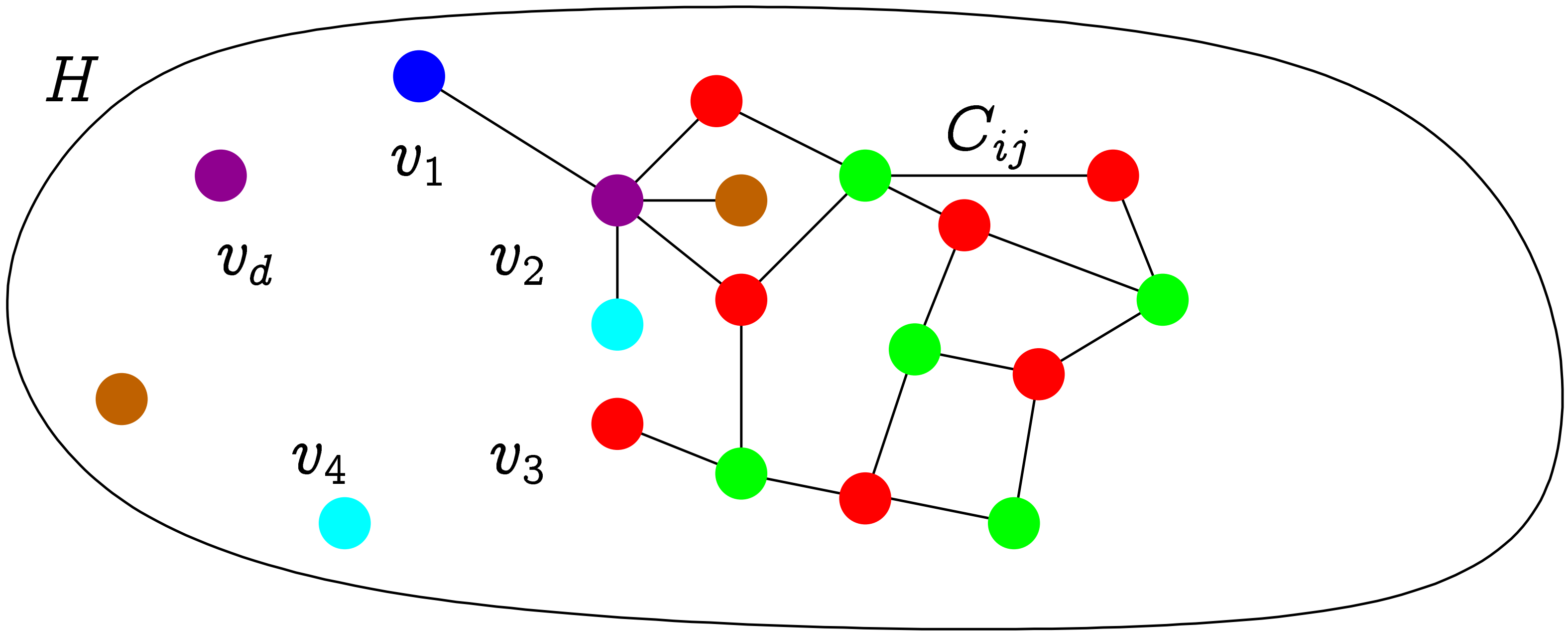
Next we will show that C_{ij} must be a path with end-vertices v_i and v_j.

It is enough to prove that otherwise H would have a coloring c' such that the vertices v_i and v_j are in different connected components of H'_{ij}.

The degree of v_j in the graph H is $\leq d-1$. If v_j had two neighbours colored the same, the neighbours of v_j would have at most $d-2$ different colors.



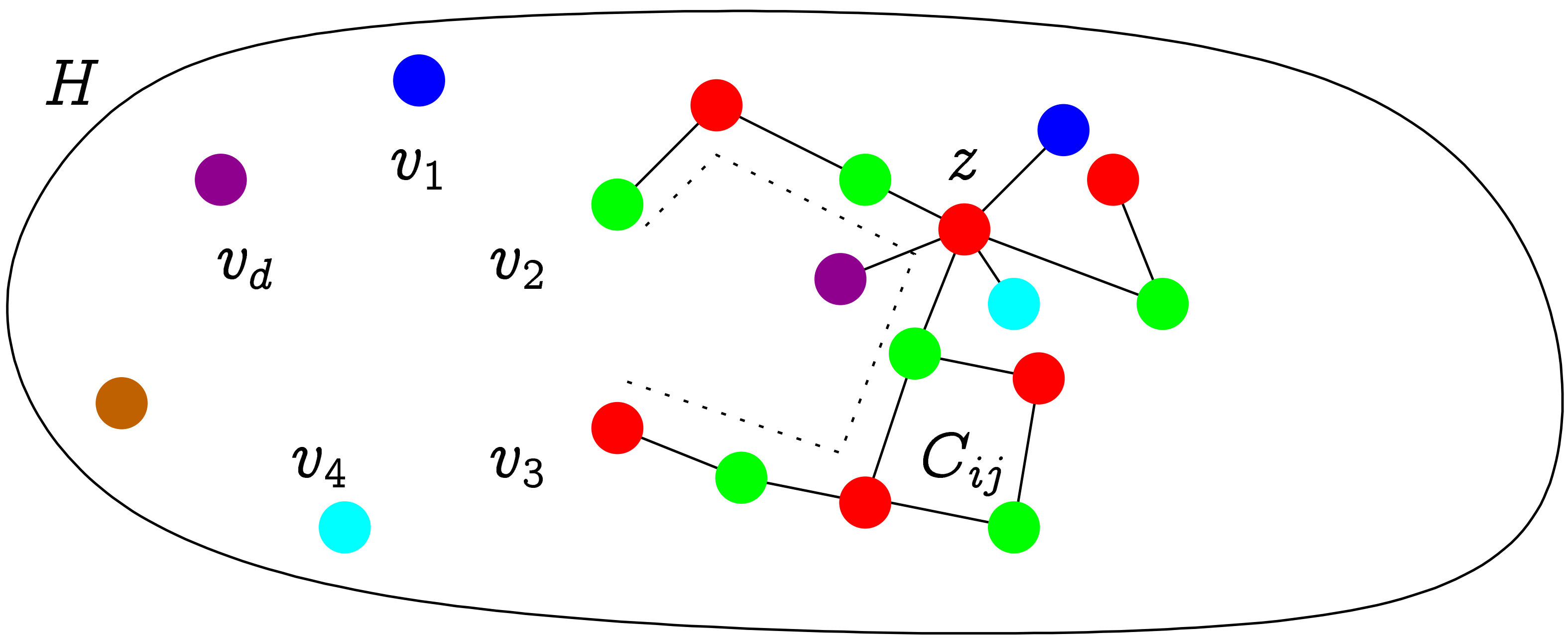
Thus the vertex v_j can be colored in two different colors. One of them is j, but thete mus be another color as well.



Now two vertices of the set $\{v_1, \ldots, v_d\}$ have the same color.

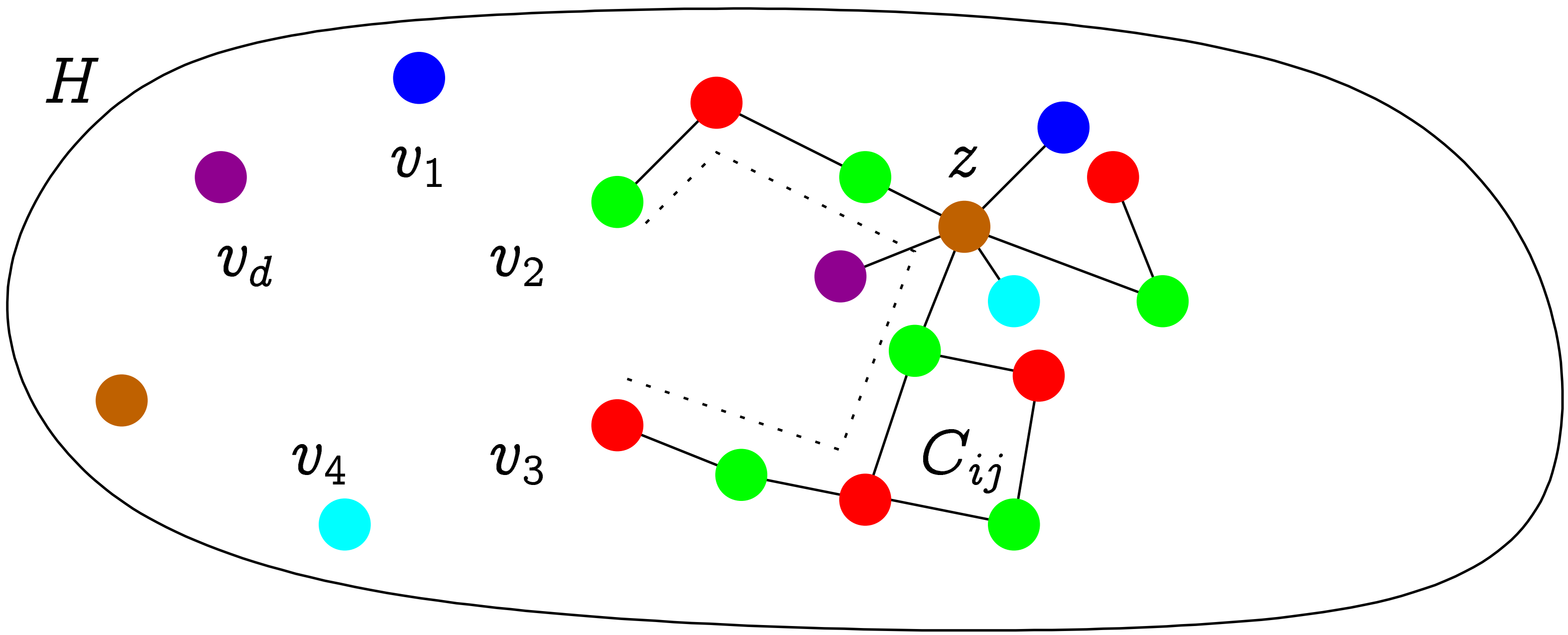
Thus $\deg_{C_{ij}}(v_j) = 1$.

Consider a path from v_j to v_i. Let z be the first vertex in this path with a degree in C_{ij} being ≥ 3.



The neighbours of z have $\leq d-2$ different colors (at most d neighbours, at least 3 colored the same).

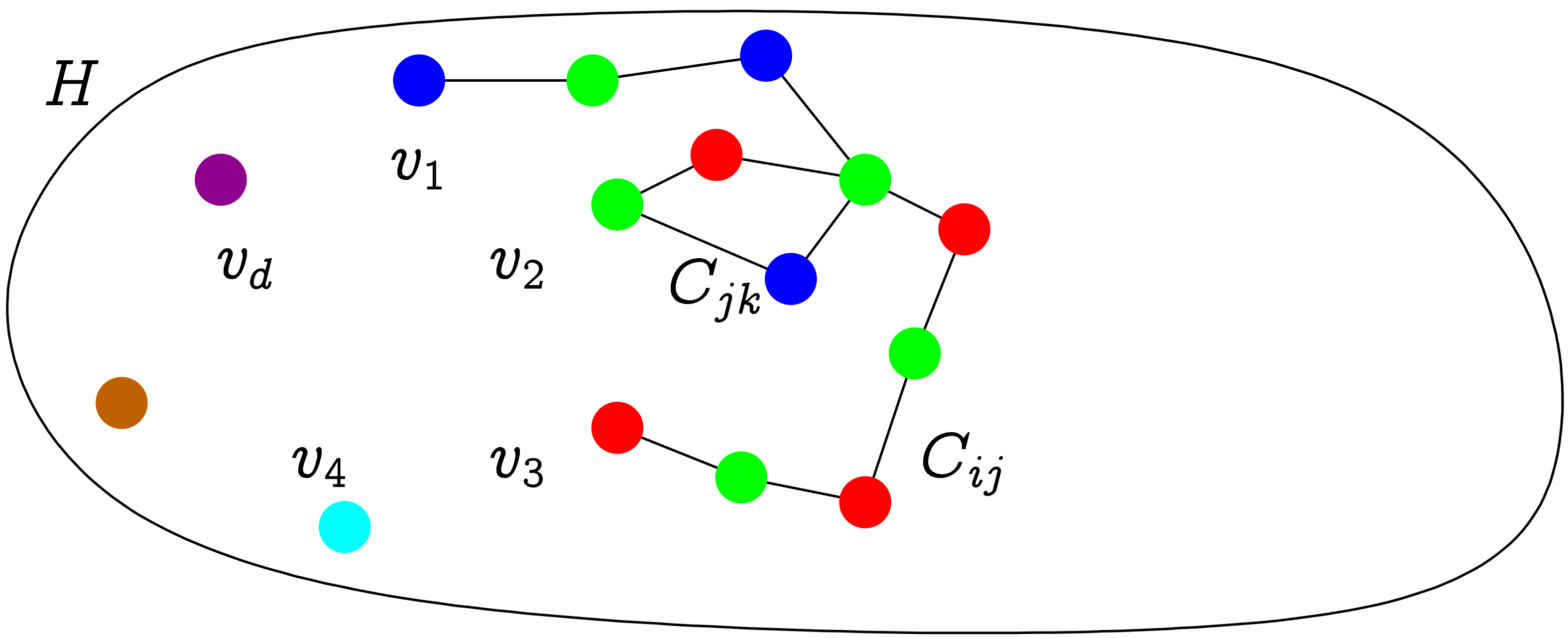
z can be colored in two ways. One of them is i or j, but there is another.



Then C_{ij} breaks into (at least) two components, v_i and v_j falling into different components.

Thus C_{ij} is a path.

Next we will show that the paths C_{ij} and C_{jk} intersect only in the vertex v_j.



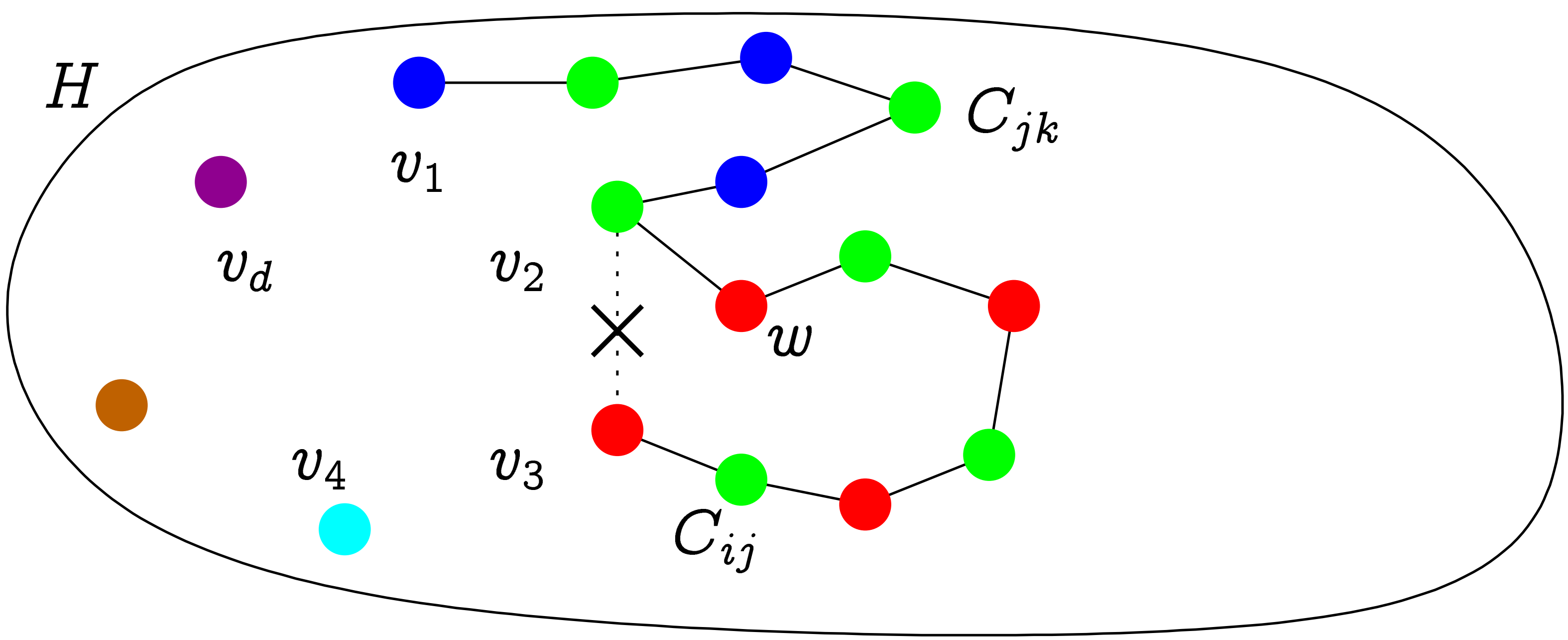
Otherwise the neighbours of the comon vertex would have at most $d-2$ different colors and and this vertex could be re-colored.

Summing up:

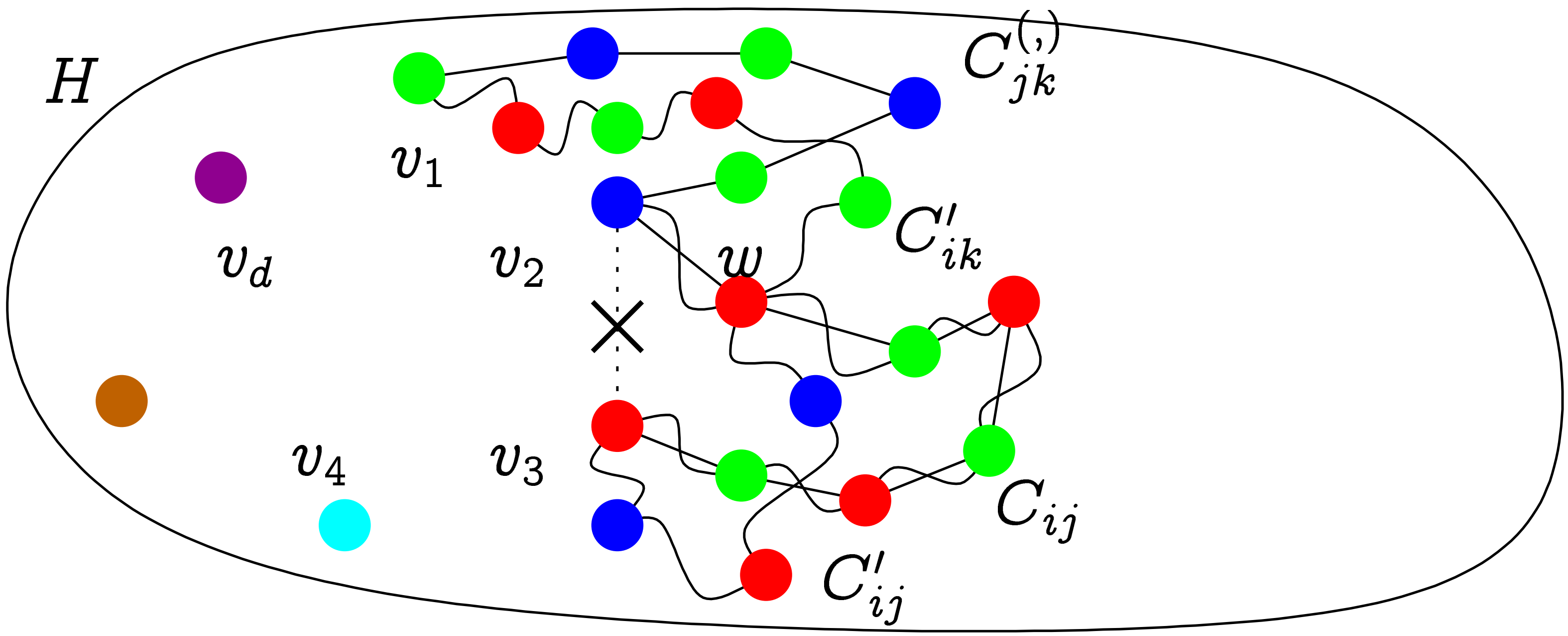
- For every two vertices v_i and v_j there is <u>exactly one</u> path C_{ij} from v_i to v_j, with the vertices being alternately of colors i and j.
  - If v_i and v_j are connected by an edge, this edge is the path.
- These paths intersect only in end-vertices.

There exist vertices v_i and v_j that are not connected by an edge (since G had no cliques of $d+1$ elements).

Let w be the neighbour of v_j being colored in i.



Exchange the colors j and k in path C_{jk}. We obtain a new coloring c' and new paths C'_{ij}.



But now w belongs to both C'_{ij} and C'_{ik}. We showed before that this leads to a contradiction. $\square$

Theorem. Planar simple graph $G = (V, E)$ is colorable with six colors.

Proof. Induction over $|V|$.

Base. $|V| = 1$. Obvious.

Step. $|V| > 1$. Let $v \in V$ be a vertew with $\deg(v) \leq 5$; such a v exists because of planarity of G. Using the induction hypothesis, $G \backslash v$ can be colored with six colors.

Let c be a coloring of $G \backslash v$ using six colors. There exists a color i such that no neighbour of v is colored with it. Thus, we can choose the color i for v. $\square$

Theorem. Planar simple graph $G = (V, E)$ is colorable with five colors.

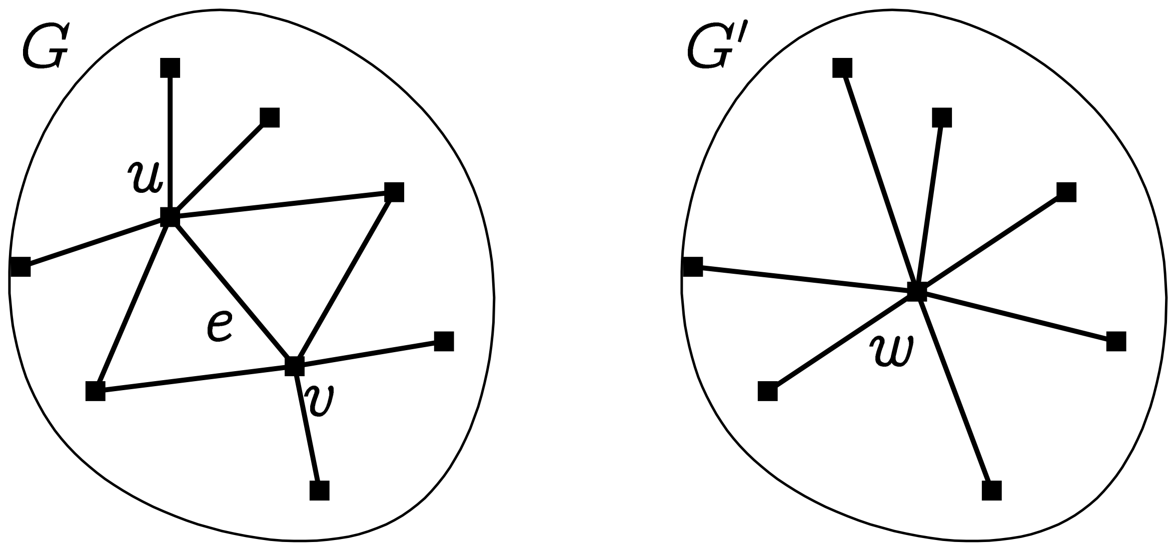
Proof. Induction over $|V|$.

Base. $|V| = 1$. Obvious.

Step. $|V| > 1$. Let $v \in V$ be a vertex with $\deg(v) \leq 5$. If $\deg(v) \leq 4$, we can use similar reasoning as in the previous theorem.

Let $\deg(v) = 5$ and $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$.

Edge *contraction* ($G \Longrightarrow G'$):



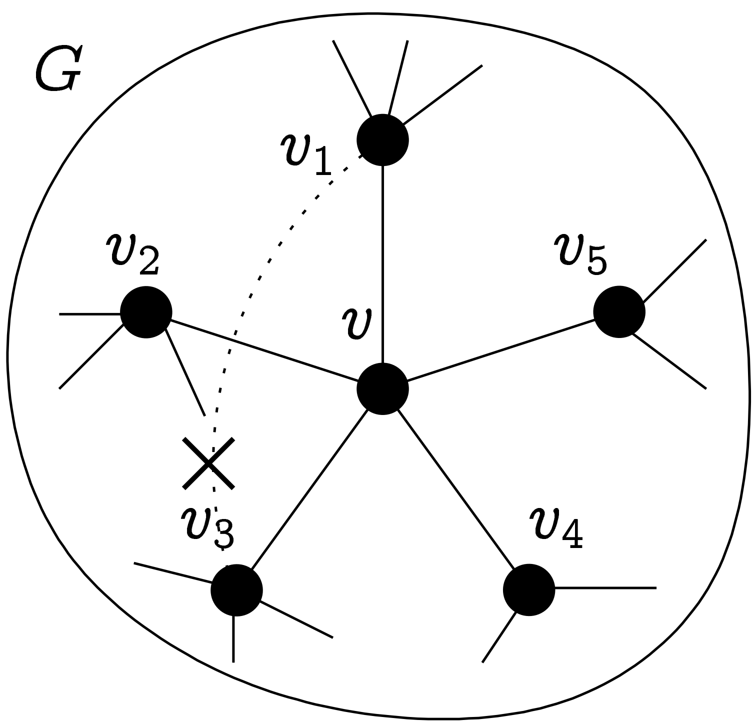
Denote G' as G/e.

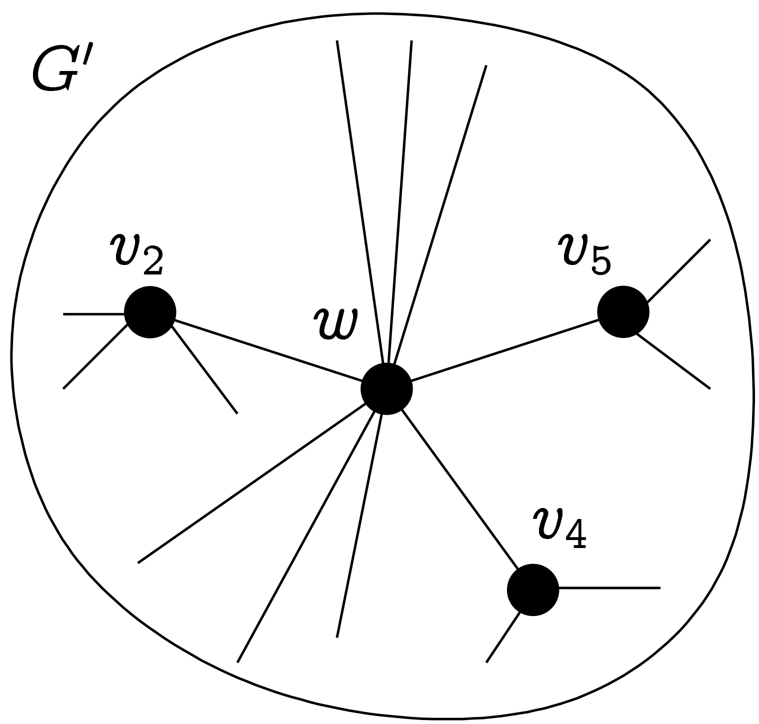
There exist $v_i, v_j \in \{v_1, v_2, v_3, v_4, v_5\}$ that are not connected by an edge.

Otherwise $K_5 \leq G$, thus G would not be planar.

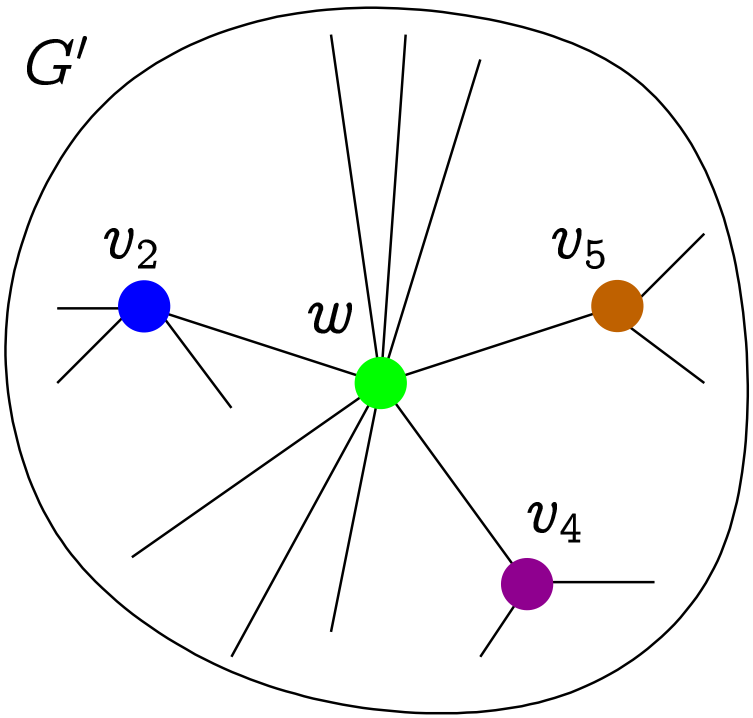
Let $G' = \big(G/(v, v_i)\big)/(v, v_j)$.

G' is like G, only instead of v, v_i, v_j we have one vertex w.

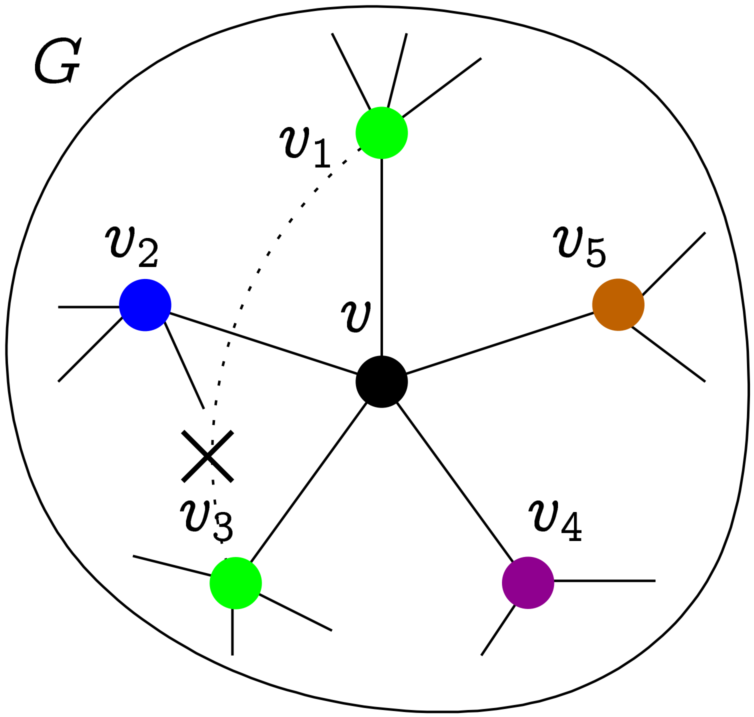




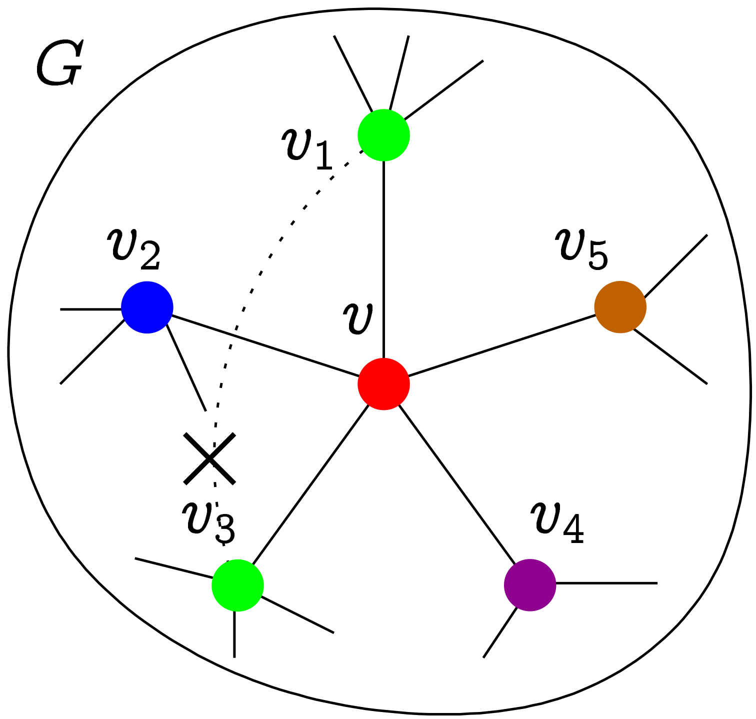
Apply induction hypothesis to G'. Let c be a coloring of G' with five colors.



The vertices v_i and v_j of G will be colored the same as w. . .



and v using the color that remained unused among its neighbours.



$\square$

Theorem. (Appel and Haken, 1976) Planar simple graph $G = (V, E)$ is colorable with four colors.

Proof. Cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet. $\square$

Let $G = (V, E)$ be a (simple) graph. In how many ways is it possible to color it using k colors?

I.e. how many functions $c : V \longrightarrow \{1, \ldots, k\}$ are there defining a correct vertex coloring with k colors?

I.e., if

- c is a coloring of G using k colors;
- $\sigma : V \longrightarrow V$ is an automorphism of G;
- $\varphi : \{1, \ldots, k\} \longrightarrow \{1, \ldots, k\}$ is a bijection,

we consider c, $c \circ \sigma$ and $\varphi \circ c$ to be different.

The number of such colorings will be denoted by $\mathrm{P}_G(k)$.

We have:

$$\mathrm{P}_{O_n}(k) = k^n$$
$$\mathrm{P}_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1)$$

If T is a tree on n vertices, then

$$\mathrm{P}_T(k) = k(k-1)^{n-1} \quad .$$

Theorem. Let $G = (V, E)$ be a simple graph. Let $e \in E$. Then $\mathrm{P}_G(k) = \mathrm{P}_{G-e}(k) - \mathrm{P}_{G/e}(k)$.

Proof. Let $u, v \in V$ be the end-vertices of e.

- The number of colorings c of the graph $G-e$, such that $c(u) \neq c(v)$, is the same as the number of colorings of the graph G.
- The number of colorings c of the graph $G-e$, such that $c(u) = c(v)$, is the same as the number of colorings of the graph G/e.

Thus $\mathrm{P}_{G-e}(k) = \mathrm{P}_G(k) + \mathrm{P}_{G/e}(k)$. $\square$

Corollary. P_G is a polynomial.

Proof. Induction over E.

Base. $G = O_n$. Then $\mathrm{P}_G(k) = k^n$.

Step. G has some edges, let e be one of them. Then $\mathrm{P}_G(k) = \mathrm{P}_{G-e}(k) - \mathrm{P}_{G/e}(k)$. By induction hypothesis, P_{G-e} and $\mathrm{P}_{G/e}$ are polynomials, making P_G as a difference of two polynomials a polynomial, too. $\square$

The function P_G is called a *cromatic polynomial* of G.

The last theorem can be used to compute chromatic polynomials of graphs.

Let a simple graph G have n vertices and m edges. Let $G_1, \ldots, G_t$ be its connected components. Then

- $\mathrm{P}_G(k)$ is a polynomial of degree n.
- The coefficient of k^n in $\mathrm{P}_G(k)$ is 1.
- The coefficient of k^{n-1} in $\mathrm{P}_G(k)$ is $-m$.
- The coefficients of $\mathrm{P}_G(k)$ have alternating signs.
- $\mathrm{P}_G(k) = \prod_{i=1}^{t} \mathrm{P}_{G_i}(k)$.
- If G is connected, then the free term of $\mathrm{P}_G(k)$ is zero and the linear term is different from zero.

Proofs proceed (mostly) by induction on E, using the last theorem and the base case $\mathrm{P}_{O_n}(k) = k^n$.

Prove as a homework.