Tutte theorem

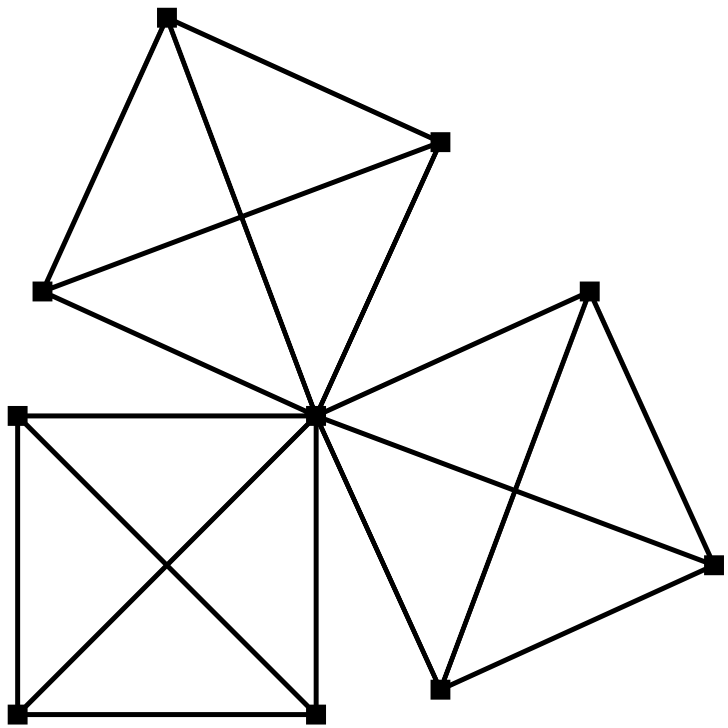
Let $G = (V, E)$ be a soimple graph. *Matching* in the graph G is a set of edges $M \subseteq E$ such that for each $v \in V$ we have $\deg_M(v) \leq 1$.

The matching M is *perfect*, if for every $v \in V$ we have $\deg_M(v) = 1$.

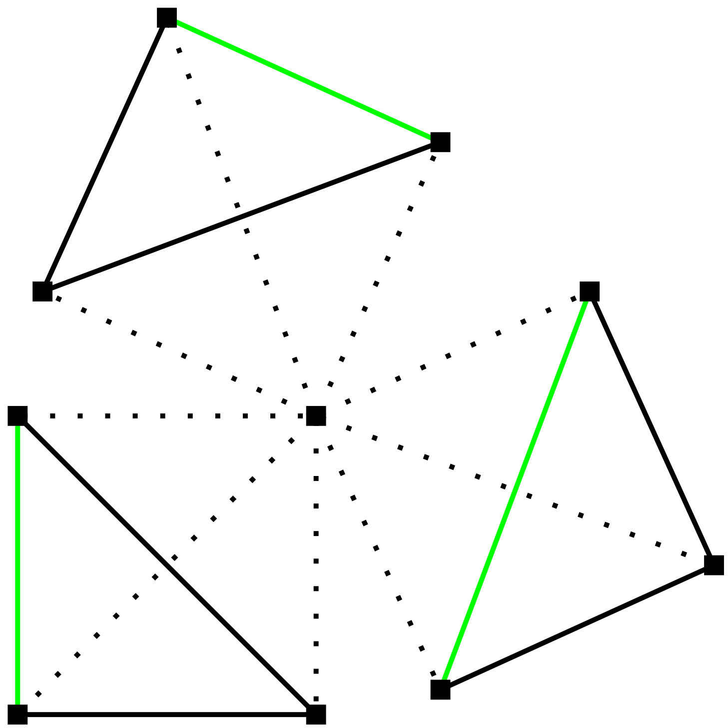
In this lecture we will give a necessary and sufficient condition for existence of a perfect matching.

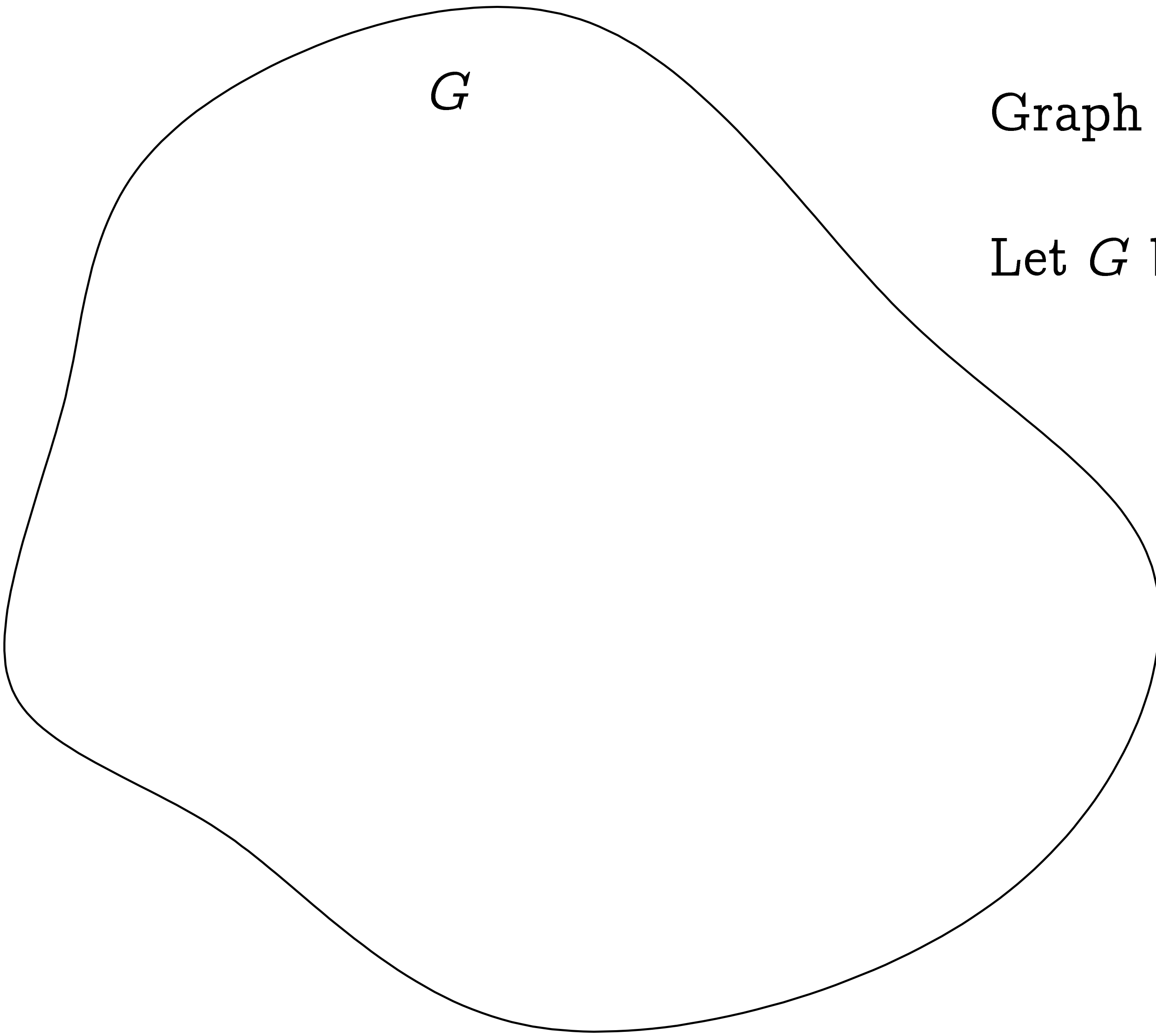
Obviously, a graph has a perfect matching iff all its connected components do. Thus it is enough to consider only connected graphs.

There is a simple necessary condition – the humber of vertices must be even.



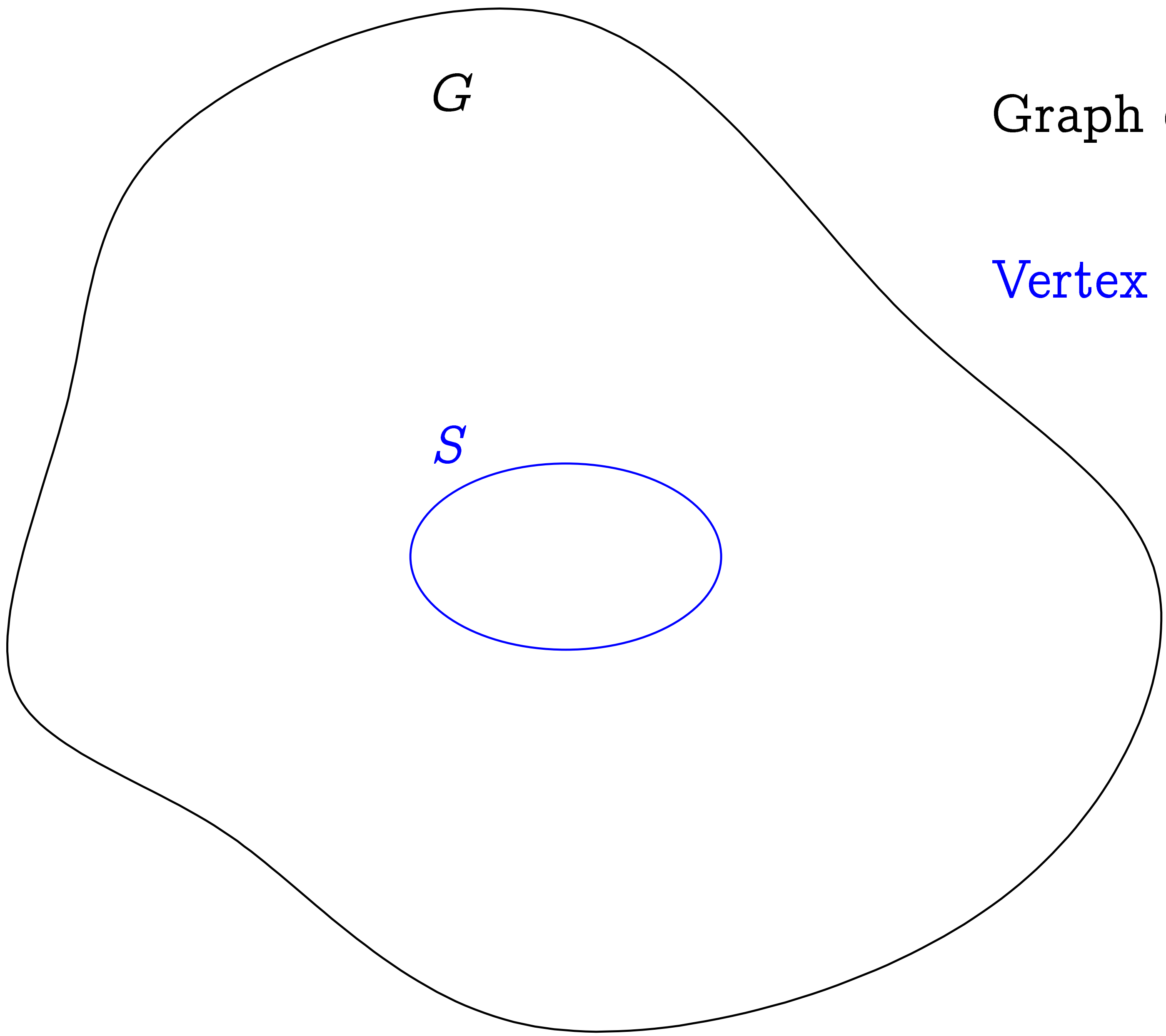
This graph has no complete matching





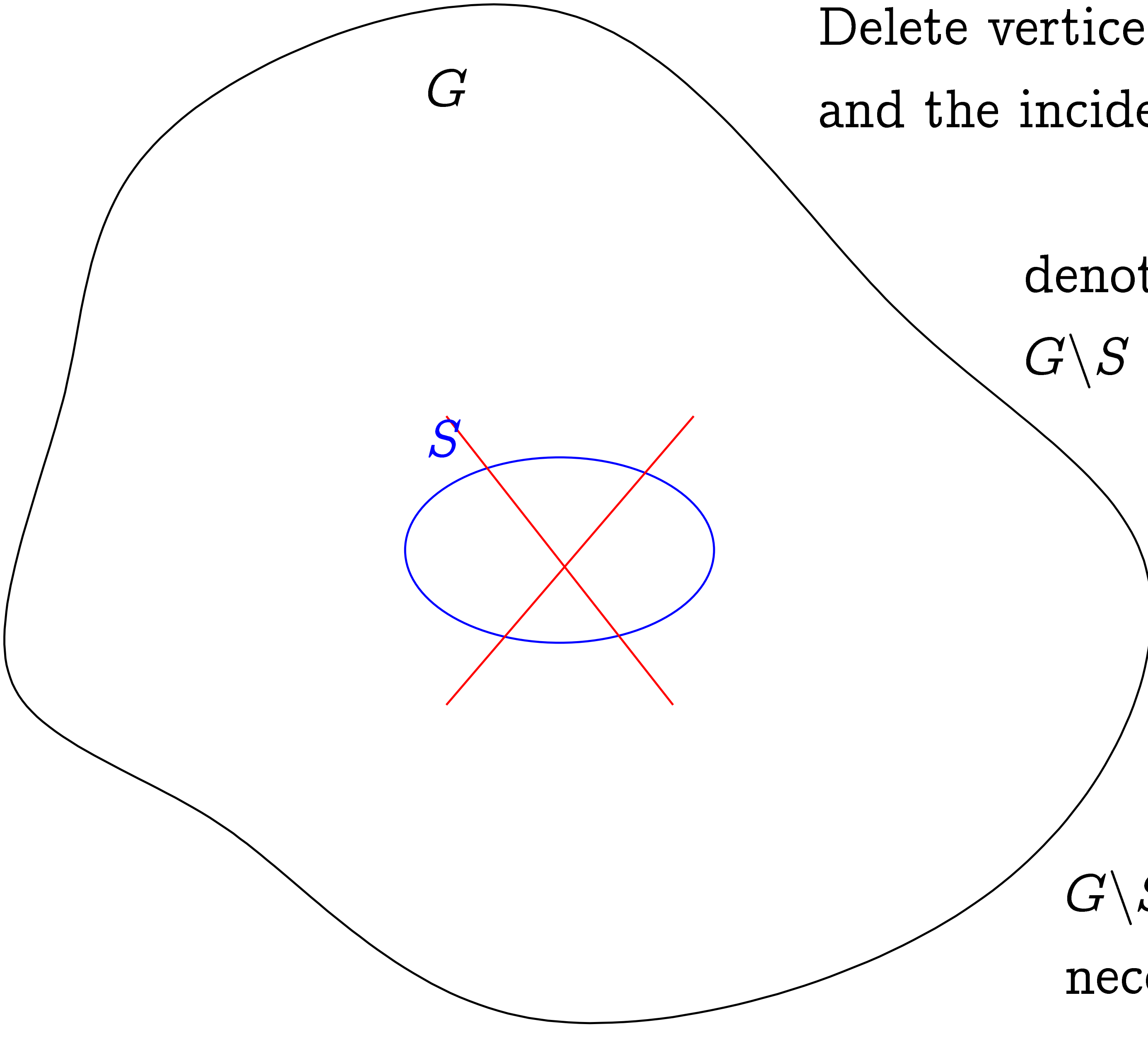
Graph $G = (V, E)$

Let G be connected



Graph $G = (V, E)$

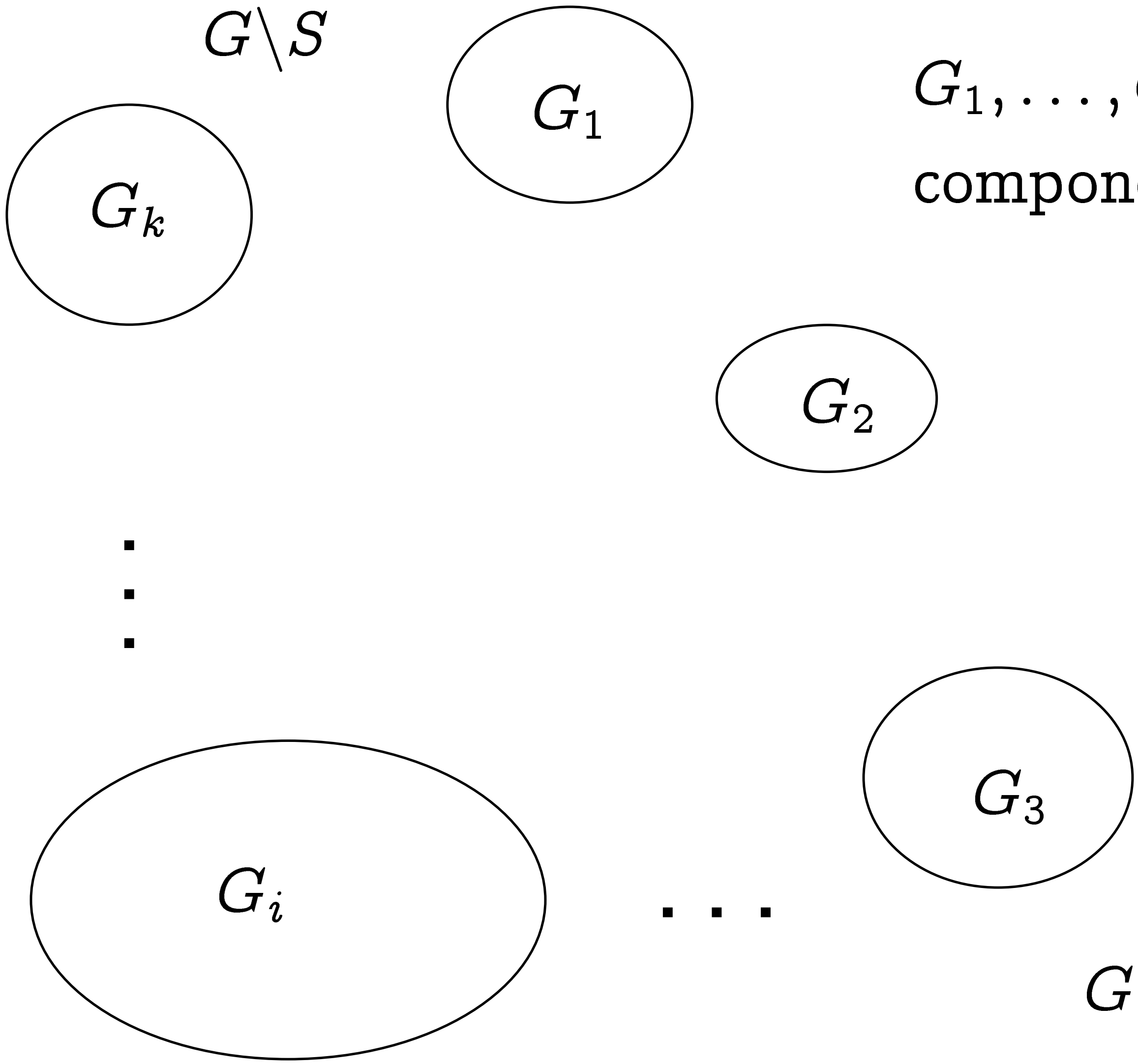
Vertex set $S \subseteq V$



Delete vertices of S
and the incident edges

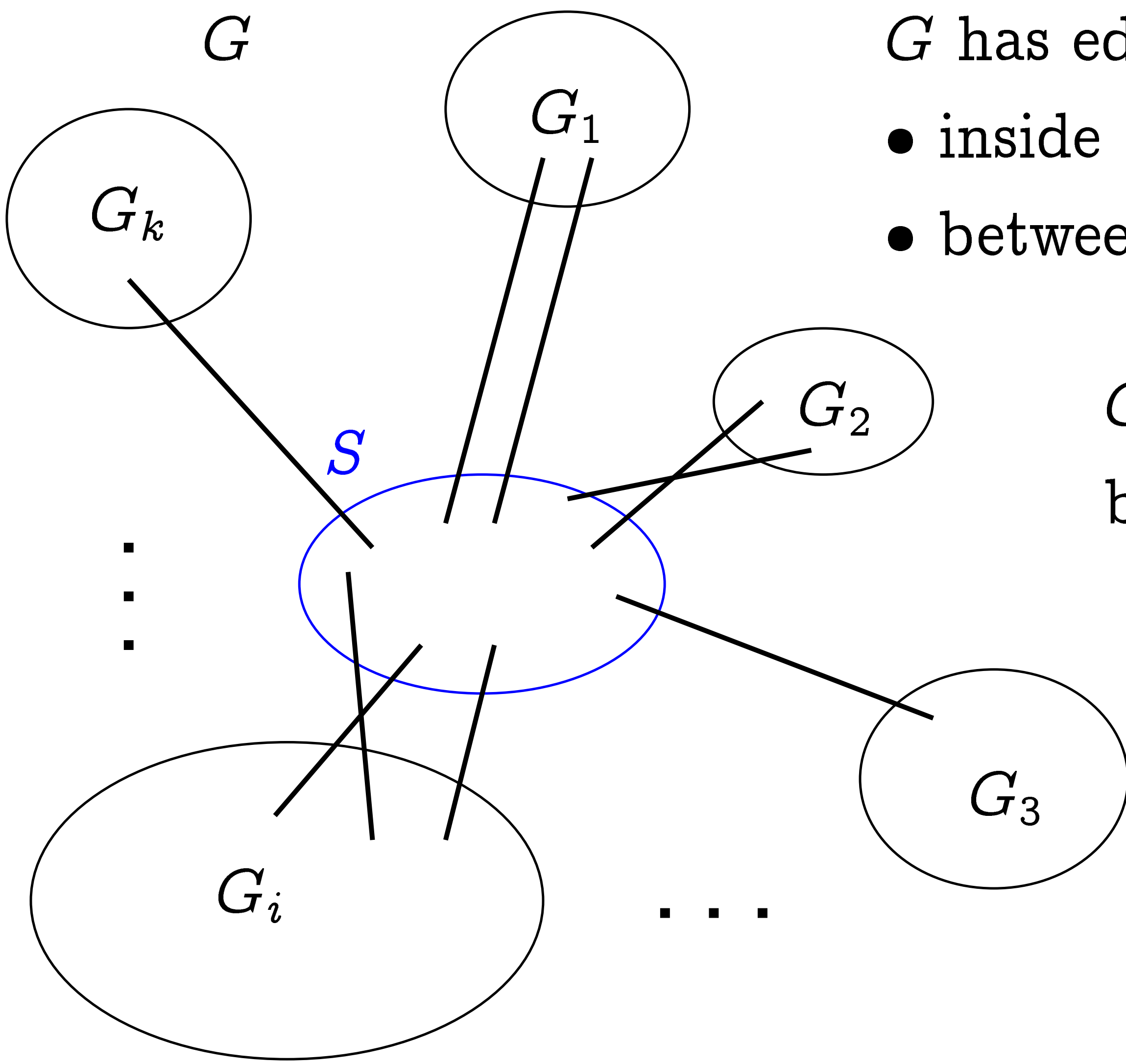
denote the resulting graph
$G\backslash S$

$G\backslash S$ is not
necessarily connected



$G_1, \ldots, G_k$:
components of $G \backslash S$

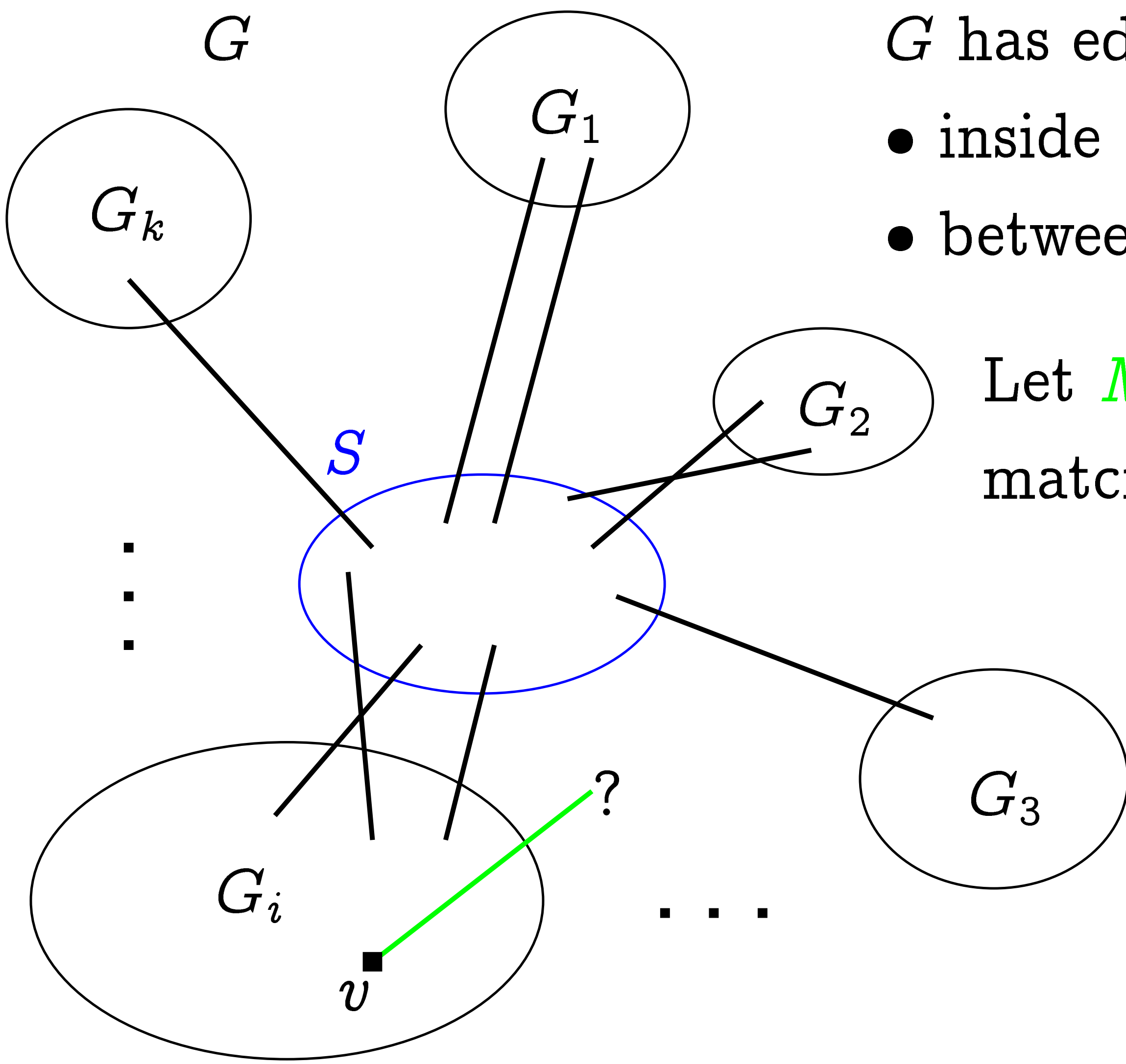
$G \backslash S$ is not
necessarily connected



G has edges:

- inside S and G_i-s
- between S and G_i-s

G has no edges between G_i-s

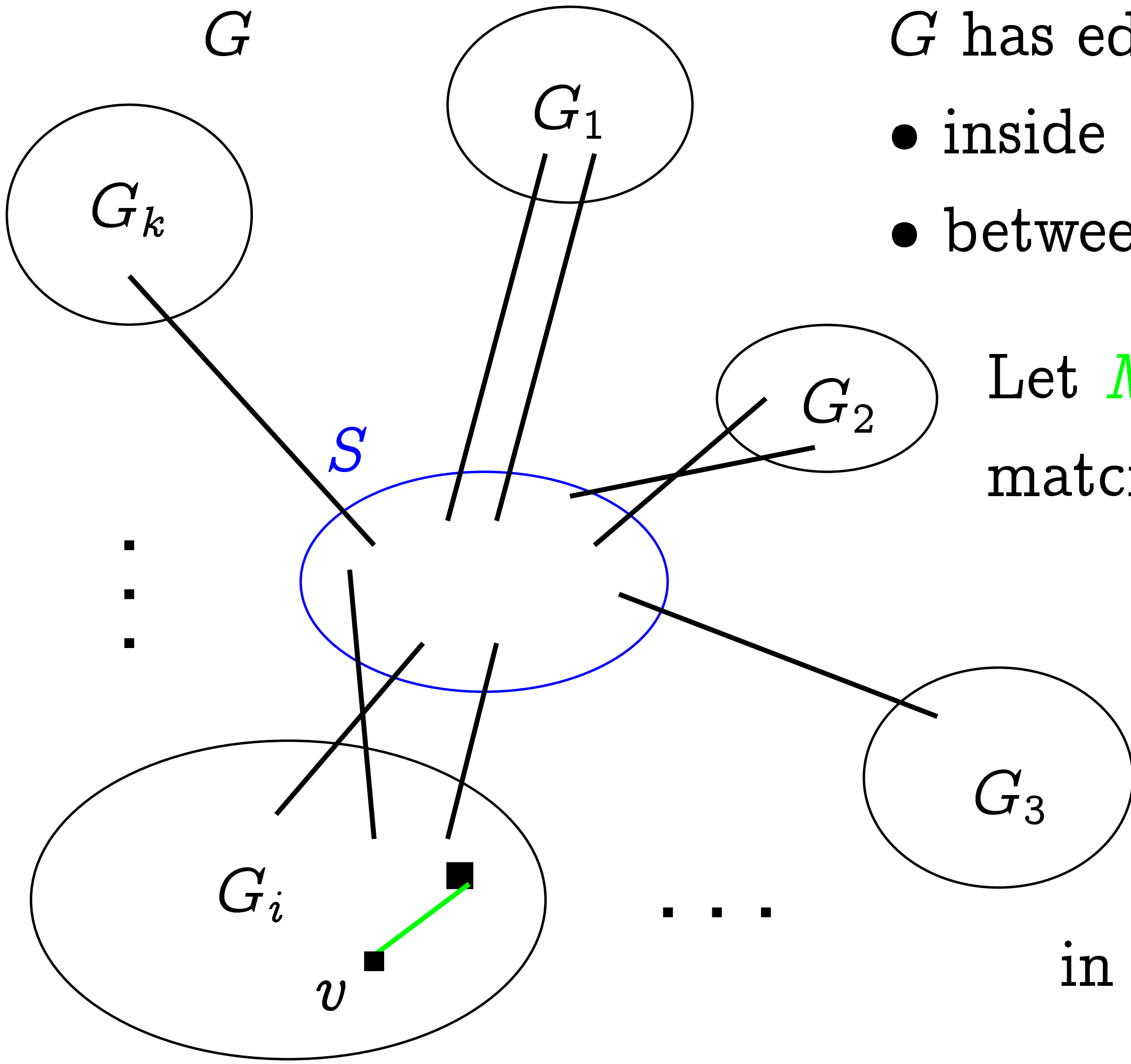


G has edges:

- inside S and G_i-s
- between S and G_i-s

Let M be a perfect matcing in G

Let v be a vertex in G_i
Where is the vertex mathced with v?



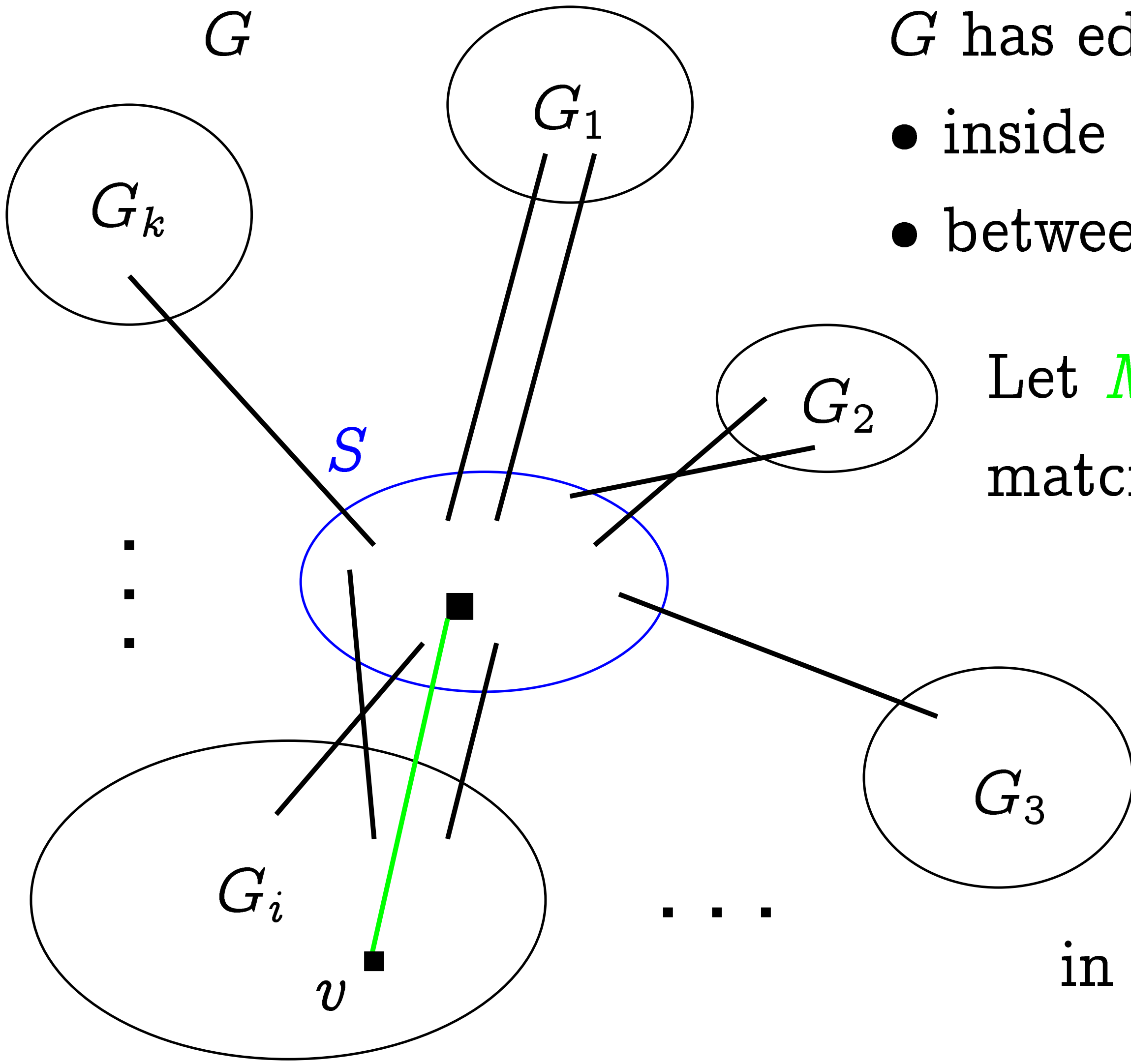
G has edges:

- inside S and G_i-s
- between S and G_i-s

Let M be a perfect matcing in G

Let v be a vertex in G_i
Where is the vertex mathced with v?

in G_i



G has edges:

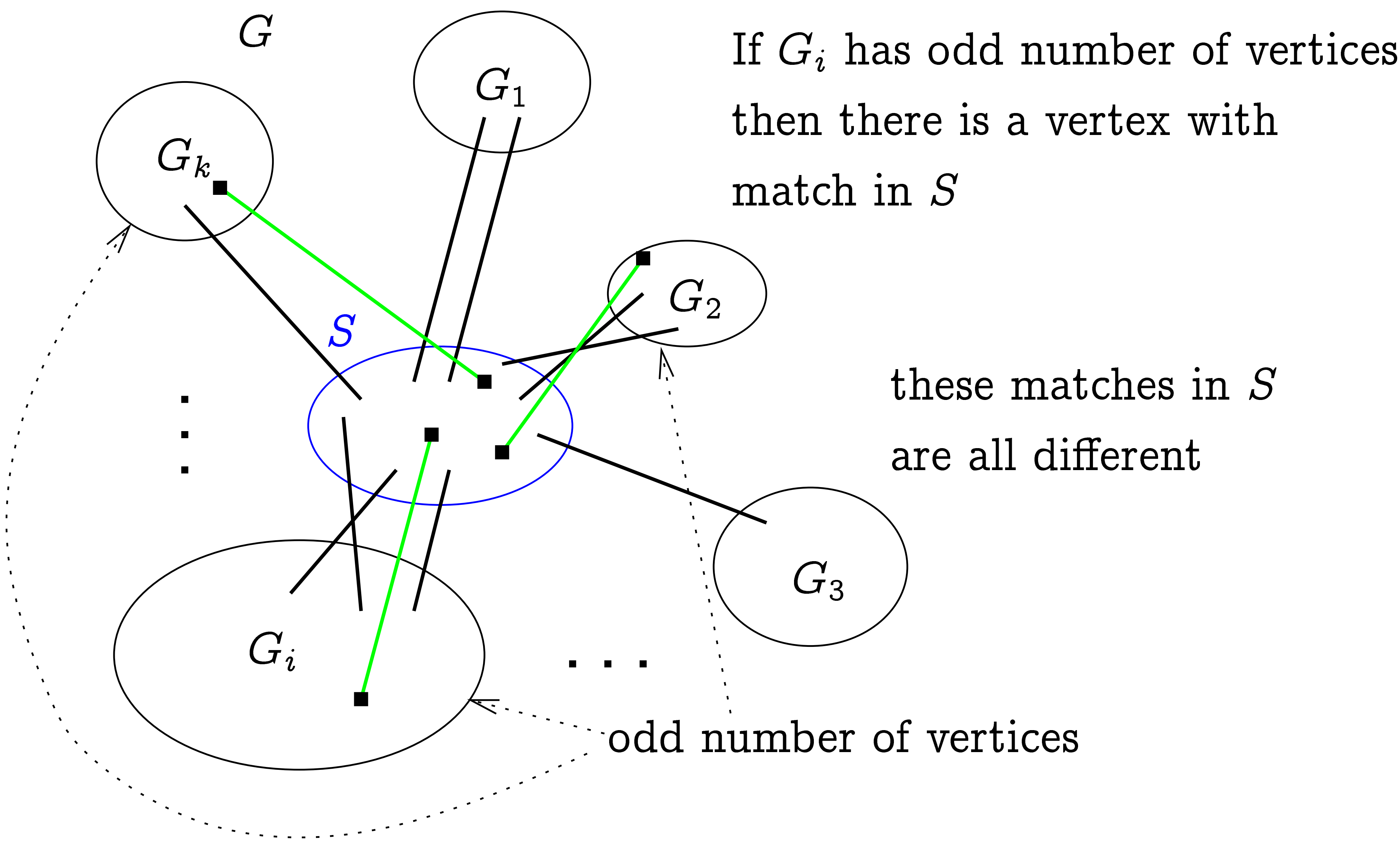
- inside S and G_i-s
- between S and G_i-s

Let M be a perfect matcing in G

Let v be a vertex in G_i

Where is the vertex mathced with v?

in S



G
G_1
G_k
G_2
S
G_3
G_i
If G_i has odd number of vertices
then there is a vertex with
match in S
these matches in S
are all different
odd number of vertices

Let $odd(G)$ denote the number of connected components in G having an odd number of vertices.

We showed that if $G = (V, E)$ has a perfect matcing, then $odd(G \backslash S) \leq |S|$ for any $S \subseteq V$.

Theorem (Tutte). Graph $G = (V, E)$ has a perfect matching iff for every $S \subseteq V$ the inequality $odd(G \backslash S) \leq |S|$ holds.

Proof. We showed necessity. Let's show sufficiency.

Assume to the contrary that there exists a graph G, such that for any $S \subseteq V$ the inequality $odd(G \backslash S) \leq |S|$ holds, but G has no perfect matching.

Note that $odd(G) = odd(G \backslash \emptyset) \leq 0$, thus G has an even number of vertices.

Add edges to G until we reach a graph G^* without a perfect matcing, but when any new edge is added, there will be a perfect matching.

Since K_{2n} has a perfect matching, such a G^* must occur.

We will show that for every $S \subseteq V$ we have $odd(G^* \backslash S) \leq |S|$.

It is enough to prove $odd(G^*\backslash S) \leq odd(G\backslash S)$.

Graph $G^*\backslash S$ is obtained by adding edges to $G\backslash S$. How has $odd(\cdot)$ changed in the process?

Adding an edge may connect 2 vertices in

- the same connected component. $odd(\cdot)$ does not change.
- different connected components. Then two components become one.

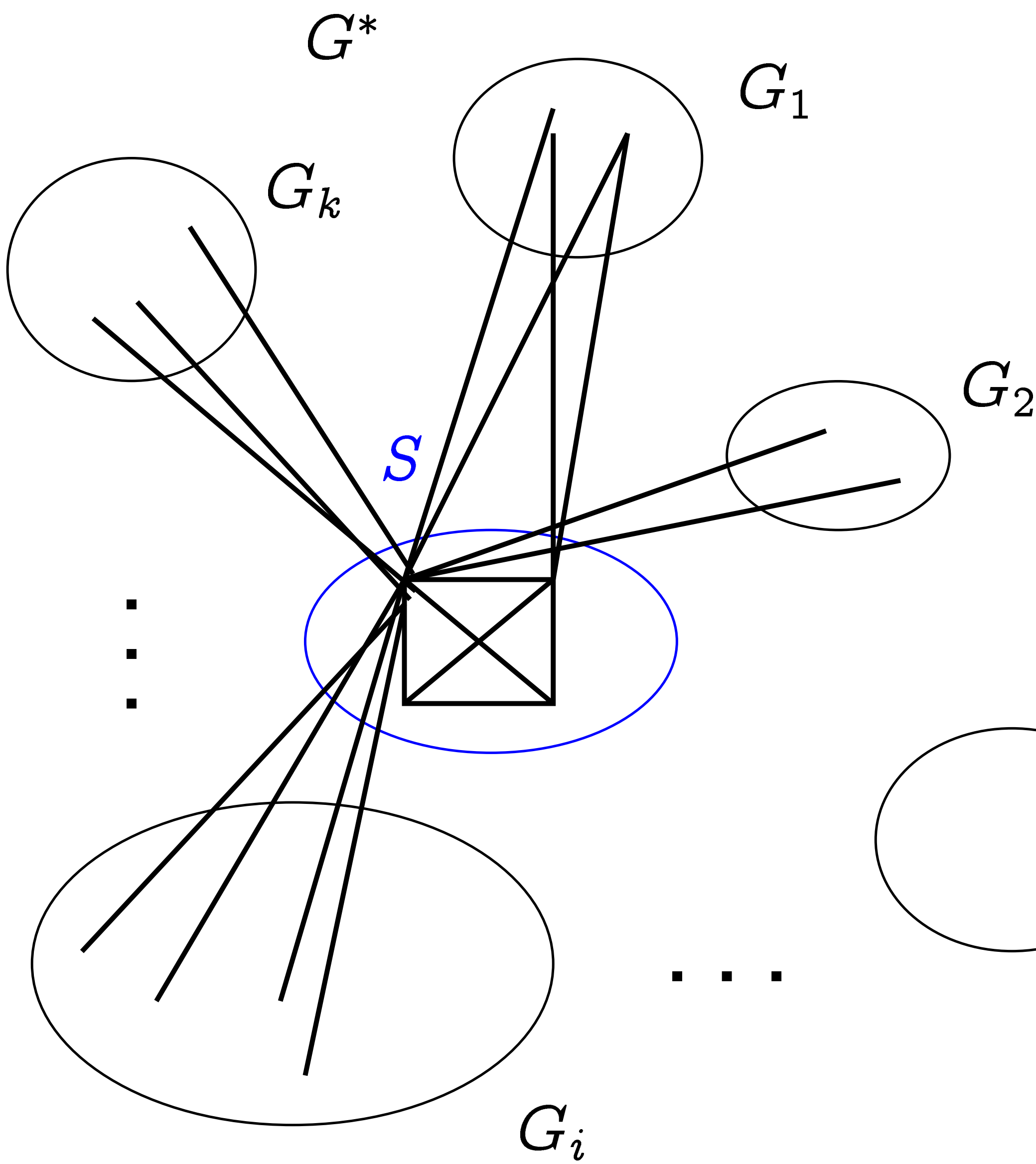
- If both components had an odd number of vertices, then the new component is even. $odd(\cdot)$ does not change.
- If one of the components was even and the other one odd, then the new component is odd. $odd(\cdot)$ does not change.
- If both components were odd, the new component is even. $odd(\cdot)$ decreases by 2.

Thus, when edges are added, $odd(\cdot)$ can only decrease. Thus $odd(G^*\backslash S) \leq odd(G\backslash S) \leq |S|$.

We have shown that the proof will follow, if we can get a contradiction from the following:

There is a graph $G^* = (V, E^*)$, such that

- it has no perfect matching;
- adding any edge will create a perfect matching;
- for any $S \subseteq V$ we have $odd(G^* \backslash S) \leq |S|$.



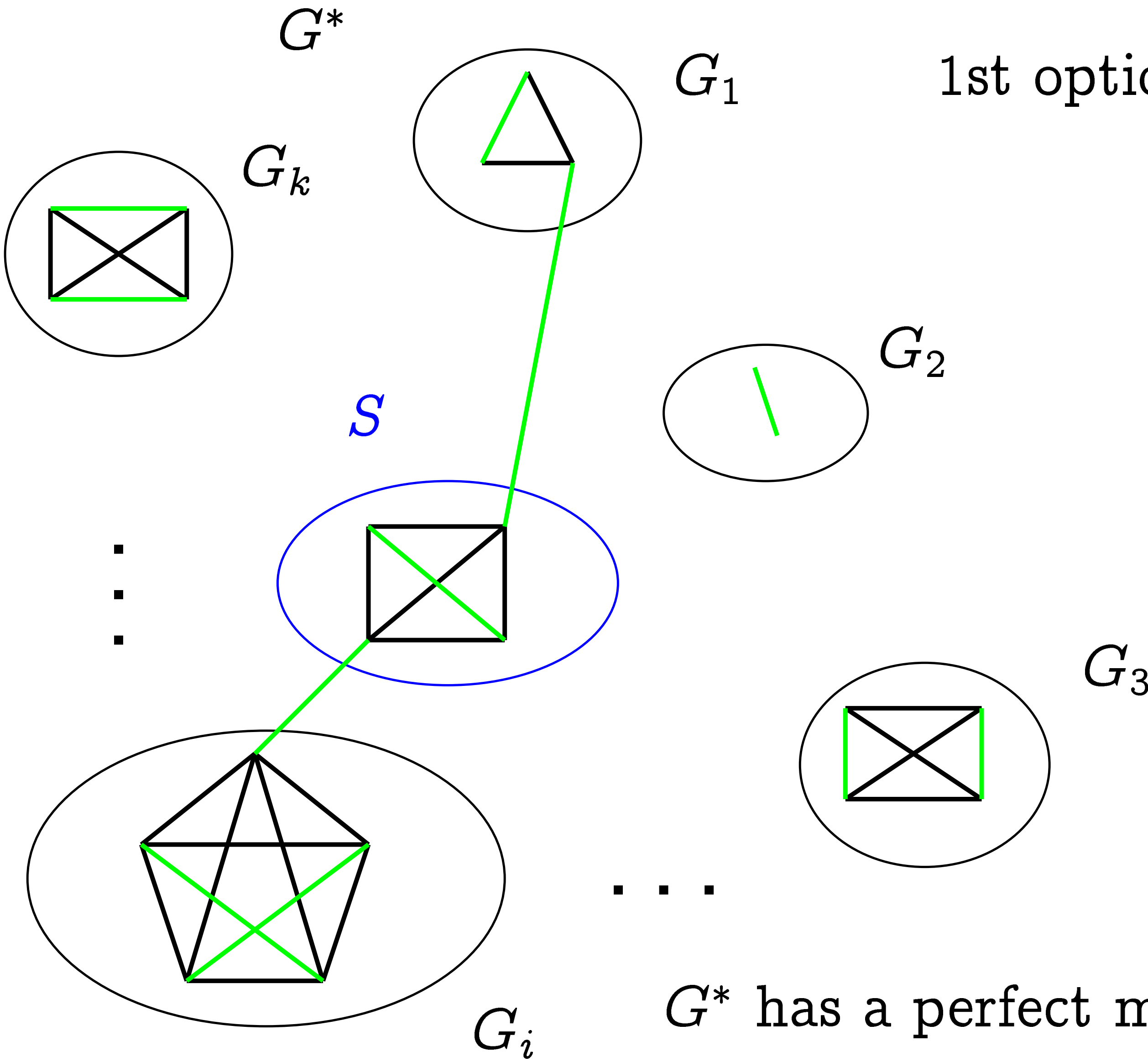
Let S be the set of all the vertices being connected to all the other vertices

There are two options:

1. All the connected components of $G^* \setminus S$ are complete graphs.

2. There exists a connected component of $G^* \setminus S$ that is not complete

1st option

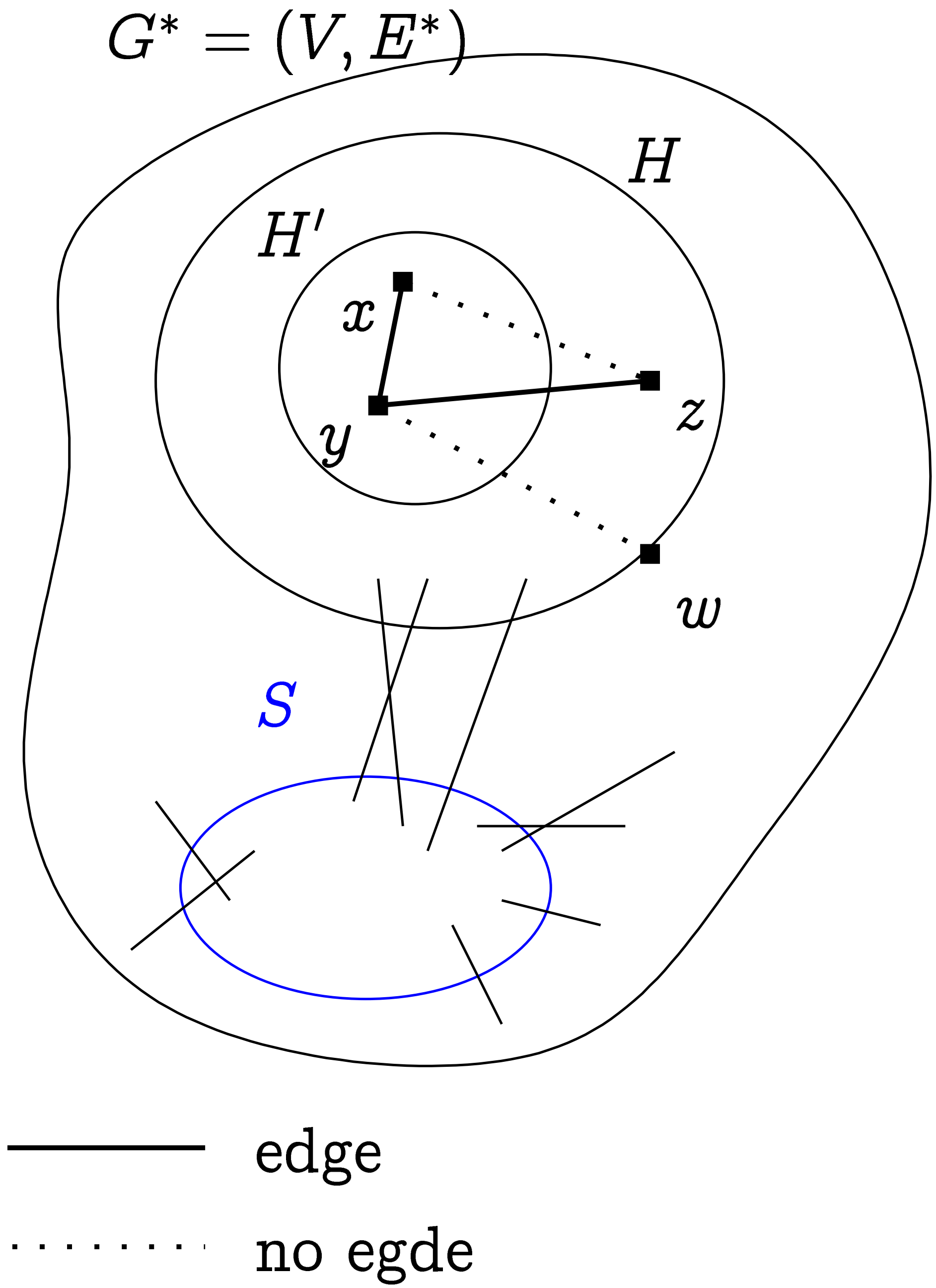


G^* has a perfect matching – a contradiction

Perfect matching in G^*-s:

- in connected components K_{2n} of $G^* \backslash S$ – within the components.
- in connected components K_{2n+1} of $G^* \backslash S$ – within the components so that one vertex is left over.
- the left-over vertices of components K_{2n+1} will be matched with vertices of S.

  There are no more components K_{2n+1} than $|S|$.
- the remaining vertices of S will be matched to each other. There is an even number of remaining vertices, since the number of vertices in G^* is even.



2nd option

H — component of $G^* \backslash S$
H is not complete

H' — H max. compl. subgraph

$y \in V(H')$ and $z \in V(H) \backslash V(H')$

$x \in V(H')$

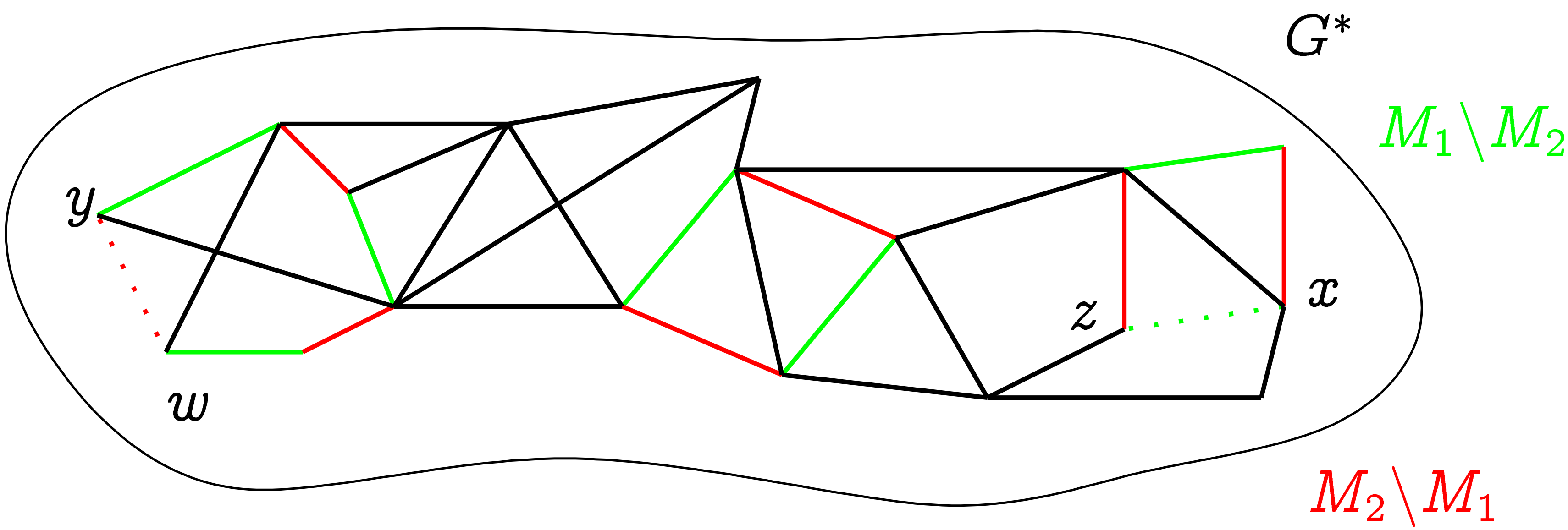
$w \in V \backslash S$

$G_1 = (V, E^* \cup \{(x, z)\})$
$G_2 = (V, E^* \cup \{(y, w)\})$
Graphs G_1 and G_2 have perfect matchings

Let M_1 be a perfect matching in G_1. Then $(x, z) \in M_1$, since otherwise M_1 would be a perfect matcing in G^*.

Let M_2 be a perfect matching in G_2. Then $(y, w) \in M_2$.

Let $G' = (V, (M_1 \backslash M_2) \cup (M_2 \backslash M_1))$.



Let $v \in V$. What are the possible values of $\deg_{G'}(v)$?

There is exactly one $e_1 \in M_1$ and exactly one $e_2 \in M_2$ such that e_1 and e_2 are incident with v.

- If $e_1 = e_2$, then $\deg_{G'}(v) = 0$.
- If $e_1 \neq e_2$, then $\deg_{G'}(v) = 2$.

Thus the components of G' are isolated vertices and cycles.

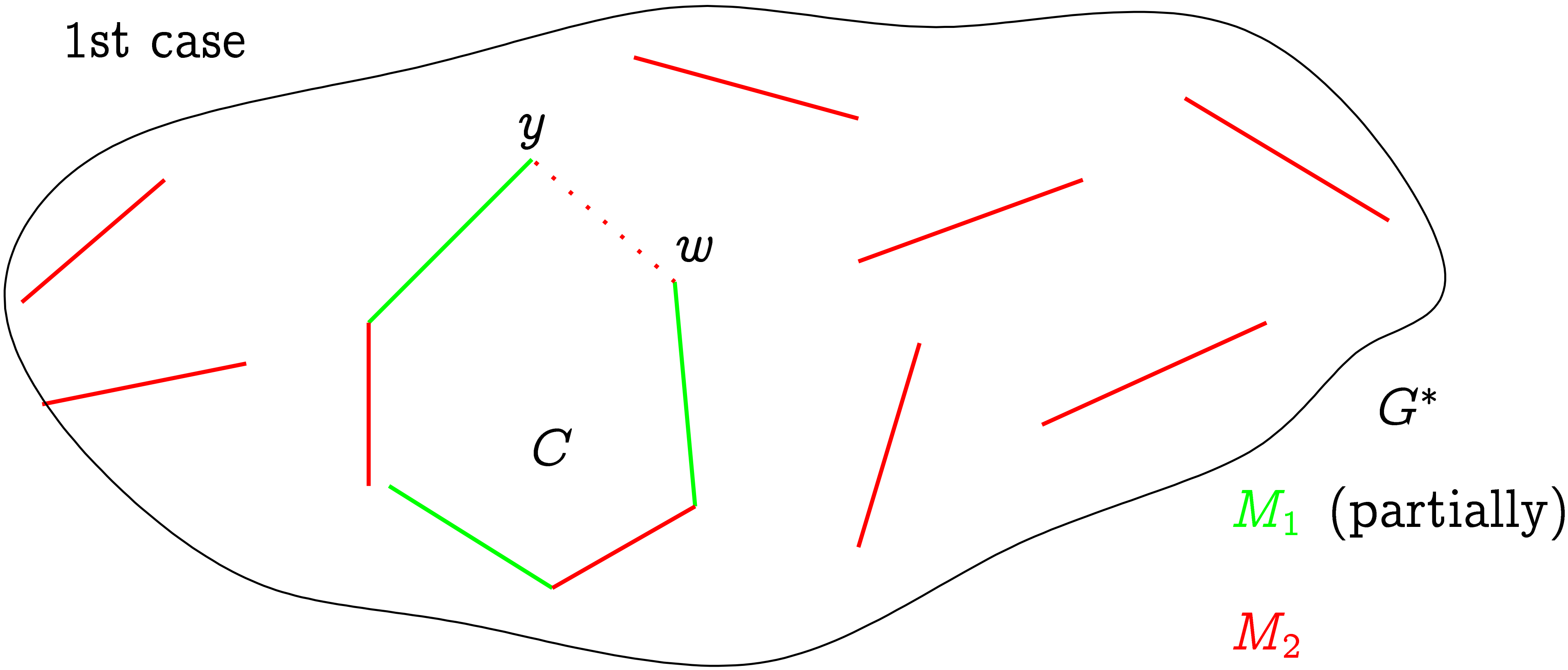
The cycles have an even length – the edges of M_1 and M_2 alternate.

There are two cases:

1. The edges (x, z) and (y, w) belong to different components of G'.

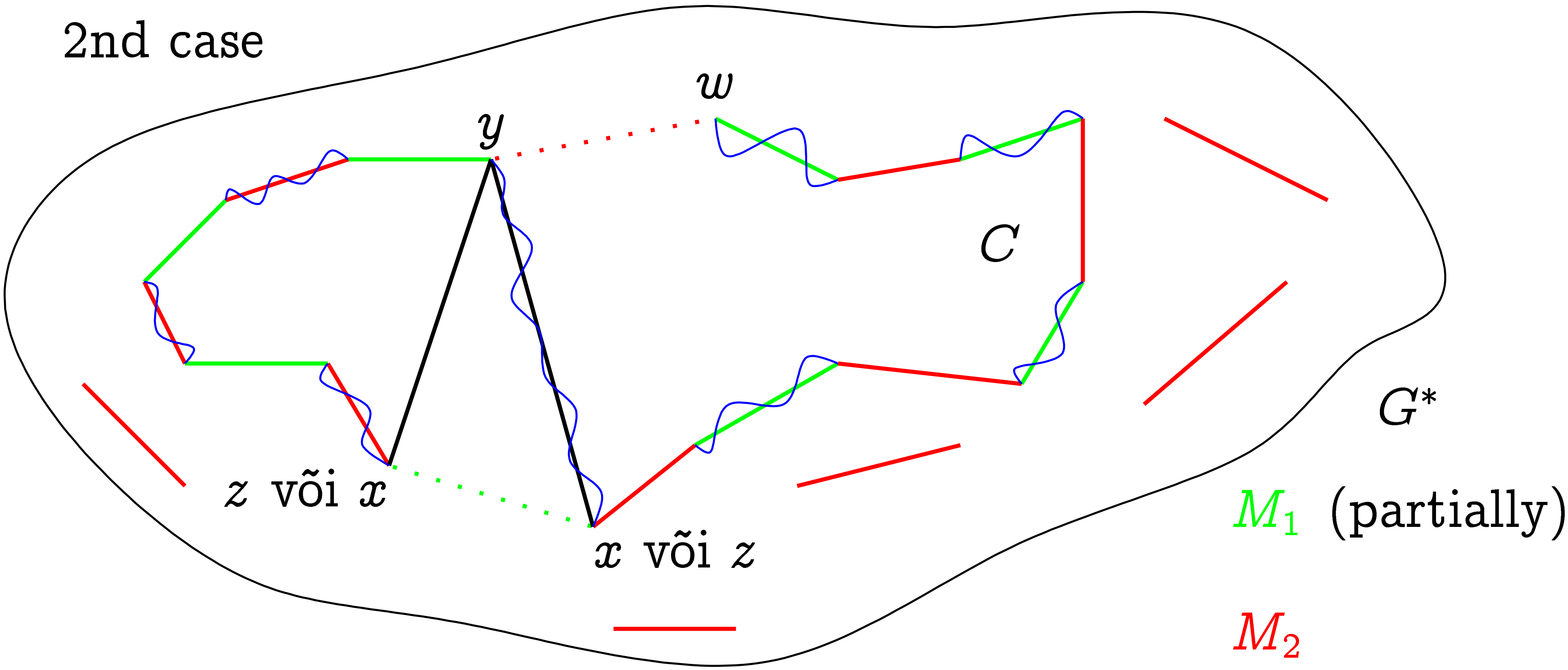
2. The edges (x, z) and (y, w) belong to the same component of G'.

We will construct a perfect matching in G^* in both cases.



Perfect matching in G^*:

- M_1 in cycle C
- M_2 outside the cycle C



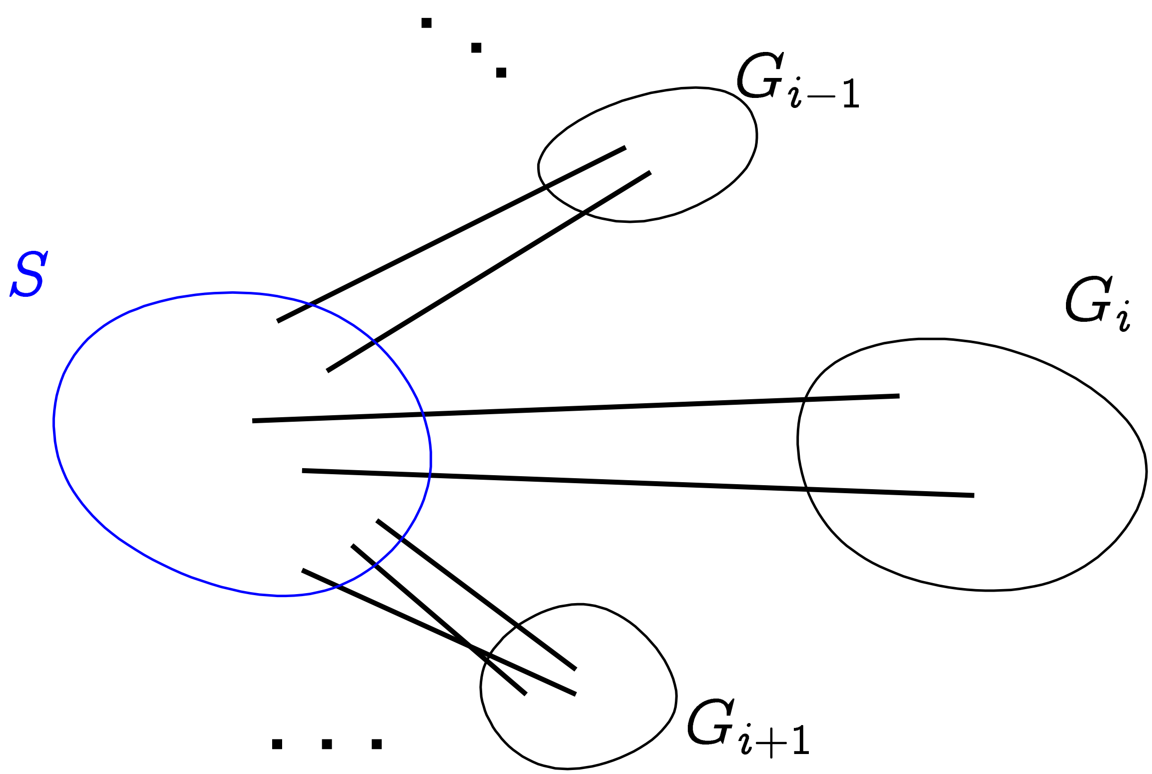
Perfect matching in G^*:

- blue edges in cycle C
- M_2 outside the cycle C $\square$

Corollary. In any 3-regular graph without bridges there is a perfect matching.

Proof. We will show that if $G = (V, E)$ is such a graph, then for every $S \subseteq V$ we have $odd(G \backslash S) \leq |S|$.

Let $G_1, \ldots, G_k$ be the connected components of graph $G \backslash S$.



Each G_i is connected to S with at least two edges, since there are no bridges.

If $|V(G_i)|$ is odd, then the number of edges with one end in G_i and another one somewhere else, is odd. (Since there is an even number of vertices with odd degree in any graph.)

All the degrees of vertices in G_i are odd, thus the number of edge ends outside G_i must be odd as well.

Thus a G_i with odd number of vertices is connected to S by at least three edges.

Let d_i be the number of edges with one end in S and another one in G_i.

Let $I \subseteq \{1, \ldots, k\}$ be the set of indices such that $i \in I$ iff $|V(G_i)|$ is odd. Then $|I| = odd(G \backslash S)$.

So we have

$$3 \cdot |S| = \sum_{v \in S} \deg(v) \geq \sum_{i=1}^{k} d_i \geq \sum_{i \in I} d_i \geq \sum_{i \in I} 3 = 3 \cdot odd(G \backslash S)$$

Thus $|S| \geq odd(G \backslash S)$ for any $S \subseteq V$. Tutte theorem implies the existence of a perfect matching. $\square$