Given a structure (graph, matrix, p.o. set), we define two kinds of objects on it:

1st kind — they connect something.
2nd kind — they separate something, or cover something.

Theorems state that the size of the smallest objects of the second kind is equal to the number of “mutually independent” objects of the first kind.

The proofs consist of two parts, “≤” and “≥”.

- One part is usually much easier than the other one.
- The following lecture slides contain only proofs (sketches) for the hard directions.
Recall: Flows and cuts

- **Network** — directed graph \( G = (V, E) \) with a source \( s \in V \) and a sink \( t \in V \), and a mapping \( \psi : E \to \mathbb{R}_+ \).
- **Flow on** \((G, \psi)\) — mapping \( \varphi : E \to \mathbb{R}_+ \), such that
  - \( \forall e \in E : \varphi(e) \leq \psi(e) \);
  - \( \forall v \in V \setminus \{s, t\} : \deg_{\varphi}(v) = \deg_{\psi}(v) \).
- **Value of flow** \( \varphi \) is equal to \( \deg_{\varphi}(s) \) or \( \deg_{\psi}(t) \).
- **Cut in** \((G, \psi)\) — a set of edges \( E' \subseteq E \), such that all paths from \( s \) to \( t \) use some edge in \( E' \).
- **Value of cut** \( E' \) is \( \sum_{e \in E'} \psi(e) \).

**Theorem (Ford and Fulkerson, 1962)**

*In a network, maximum value of flows and minimum value of cuts are equal.*
Let $M$ be a $m \times n$ matrix with 0/1 entries.

- A **line** of a matrix is its row or column.

- A **partial transversal** is a selection of entries “1” in $M$, such that no two of them lie on the same line.

- The **term rank** of $M$ is the maximum size of its partial transversals.

- A **cover** of $M$ is a set of its lines that contain all 1-s in $M$.

**Theorem (König-Egerváry, 1931)**

*The term rank of a 0/1-matrix $M$ is equal to the size of its minimum covers.*
Easy direction: minimum cover must be at least the term rank. In the other direction...

Let $c$ be the term rank of $M$.

Define a network as follows:

- Vertices: $x_1, \ldots, x_m, y_1, \ldots, y_n, s, t$.
- Edges: $s \xrightarrow{1} x_i; y_j \xrightarrow{1} t; x_i \xrightarrow{c+1} y_j$ (if $M_{ij} = 1$)

In a maximal flow $\varphi$ we have $\forall e: \varphi(e) \in \{0, 1\}$.

The value of $\varphi$ is $c$.

In a corresponding minimum cut, there are edges $s \xrightarrow{1} x_{i_1}, \ldots, s \xrightarrow{1} x_{i_k}, y_{j_1} \xrightarrow{1} t, \ldots, y_{j_l} \xrightarrow{1} t$, where $k + l = c$.

These select $k$ rows and $l$ columns that cover $M$. 

Peeter Laud (Cybernetica)
Matchings and coverings

Let $G = (V, E)$ be a simple graph.

Definition

A matching (kuuskõla) is a set $M \subseteq E$, such that $\forall v \in V : \deg_M(v) \leq 1$.

Definition

A covering (kate) is a set $V \subseteq V$, such that each edge in $E$ has at least one end-point in $C$. 
For $S \subseteq V$, let $N(S) \subseteq V$ denote the neighbourhood of $S$ — the set of vertices that neighbour at least one vertex in $S$.
Let maximum matching / minimum cover denote the matching(s) / cover(s) with maximum/minimum cardinality.

**Theorem (Hall’s Marriage theorem, 1935)**

Let $G = (X \cup Y, E)$ be a bipartite graph with parts $X$ and $Y$. $G$ has a matching $M$ with $\forall x \in X : \deg_M(x) = 1$ iff $\forall S \subseteq X : |N(S)| \geq |S|$.

**Theorem (König’s theorem for matrices, 1931)**

Let $G = (X \cup Y, E)$ be a bipartite graph. Maximum matchings and minimum covers in $G$ have the same cardinality.
Proof of König’s theorem

Given bipartite graph $G = (X \cup Y, E)$, consider the adjacency matrix of $G$.
- Size: $|X| \times |Y|$, rows indexed by $X$, columns indexed by $Y$.
- Entry in position $(u, v)$ equals 1 iff $(u, v) \in E$, otherwise it equals 0.

In this matrix
- Partial transversals correspond to matchings in $G$;
- Covers correspond to coverings in $G$. 
Proof of Hall’s theorem

- Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall’s criterion.
- Let $C$ be a minimal covering for $G$.
- If $C$ is a cover then $N(X \setminus C) \subseteq Y \cap C$.
- Thus $|Y \cap C| \geq |N(X \setminus C)| \geq |X \setminus C|$.
- Thus $|C| = |X \cap C| + |Y \cap C| \geq |X \cap C| + |X \setminus C| = |X|$.
- Thus there is a matching of size at least $|X|$.
Separating sets and disjoint paths

**Definition**

Let $G = (V, E)$ be a connected simple graph, $u, v \in V$, $S \subseteq V \setminus \{u, v\}$ and $F \subseteq E$.

- $S$ is a $(u, v)$-separating (vertex) set if $G \setminus S$ has no paths from $u$ to $v$.
- $F$ is a $(u, v)$-separating edge set if $G - F$ has no paths from $u$ to $v$.

Two paths from $u$ to $v$ are

- **vertex-disjoint** if their only common vertices are $u$ and $v$.
- **edge-disjoint** if they have no common edges.

**Theorem (Menger, 1929)**

*Maximum number of pairwise edge-/vertex-disjoint paths from $u$ to $v$ is equal to the cardinality of minimum $(u, v)$-separating edge/vertex sets.*

The graph can be directed or undirected, thus we have 4 theorems here.
Proof of Menger’s theorem (edges, directed)

- Turn $G$ into network with source $u$ and sink $v$.
  - Delete edges going into $u$ or going out of $v$.
  - Give the capacity 1 to each edge.

- $(u, v)$-separating edge set of size $c \equiv$ cut of value $c$.

- Integral flow of value $c \equiv c$ edge-disjoint paths from $u$ to $v$. 
Proof of Menger’s theorem (edge, undirected)

- Turn $G$ into network with source $u$ and sink $v$.
  - Edges incident to $u$ will be directed away from $u$.
  - Edges incident to $v$ will be directed towards $v$.
  - Other edges are replaced with directed edges in both directions.
  - The capacity of each edge is 1.

- $(u, v)$-separating edge set of size $c \equiv$ cut of value $c$.
- Integral flow of value $c \equiv c$ edge-disjoint paths from $u$ to $v$. 
Proof of Menger’s theorems (vertices)

Do the same as for edges, but also

- Split each vertex \( w \) (except \( u \) and \( v \)) into two: \( w_{\text{in}} \) and \( w_{\text{out}} \), connected by an edge.
- Give capacity 1 to these edges. Give large capacities to all original edges of \( G \).
Chains and antichains in partially ordered sets

Let \((P, \leq)\) be a partially ordered set.

**Definition**

- \(Q \subseteq P\) is a **chain** if \(\forall x, y \in Q : (x \leq y \lor y \leq x)\).
- \(Q \subseteq P\) is an **antichain** if \(\forall x, y \in Q : x \neq y \Rightarrow (x \not\leq y \land y \not\leq x)\).

**Theorem (Dilworth, 1947)**

> If \(m\) is the maximum cardinality of antichains in \(P\), then \(P\) can be partitioned into \(m\) chains.
Proof of Dilworth’s theorem

- Consider $|P| \times |P|$ matrix $M$, rows and columns indexed by $P$
  - The entry $(a, b)$ of $M$ equals 1 iff $a < b$
- A chain $a_1 < a_2 < \cdots < a_n$ gives us a partial transversal
  $\{(a_j, a_{j+1}) | j \in \{1, \ldots, n-1\}\}$ of size $n - 1$.
- A partition of $P$ to $k$ chains gives us a partial transversal of size $|P| - k$.
- Conversely, take the partition of $P$ to $|P|$ 1-element chains. Also take a partial transversal of size $|P| - k$.
  - Each “1” in it corresponds to a relation $a < b$ that can be used to join two chains.
  - Thus we get a partition of $P$ into $k$ chains.
- Let $m$ be minimal, such that $P$ can be partitioned into $m$ chains.
  - $|P| - m$ is the term rank of $M$
- There is a cover of $M$ by $|P| - m$ lines.
  - There are $m$ elements with corresponding row and column not in that cover.
  - These form an antichain in $P$. 

Peeter Laud (Cybernetica)
## Doubly stochastic matrices

**Definition**

A square matrix with entries from $\mathbb{R}_+$ is **doubly stochastic** if each row and each column of it sums up to 1.

**Definition**

A square matrix with entries from $\{0, 1\}$ is a **permutation matrix** if each row and each column of it contains exactly one 1.

**Definition**

A **convex combination** of objects $x_1, \ldots, x_k$ (supporting addition and multiplication with reals) is any object of the form $\lambda_1 x_1 + \cdots + \lambda_k x_k$, where $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_k = 1$.

**Theorem (Birkhoff and von Neumann, 1946)**

*Any doubly stochastic matrix can be expressed as a convex combination of permutation matrices.*
Let $M$ be a doubly stochastic matrix of size $n \times n$.

Consider a bipartite graph $G = (X \cup Y, E)$, where

- $X = Y = \{1, \ldots, n\}$
- There is an edge from $i \in X$ to $j \in Y$ iff $M_{ij} \neq 0$.

$G$ satisfies Hall’s criterion.

- The entries in rows from any subset $S \subseteq X$ sum up to $|S|$. It takes at least $|S|$ columns to contain these entries.

A matching covering all of $X$ gives us a permutation $\sigma$, such that $(i, \sigma(i))$ is a non-zero entry of $M$ for all $i$.

- Let $\Sigma$ be the permutation matrix corresponding to $\sigma$.

Let $\varepsilon$ be the minimum of these entries.

$$M = \varepsilon \cdot \Sigma + (1 - \varepsilon) \cdot M',$$ where $M'$ is a doubly stochastic matrix with at least one more zero entry.

We can do induction over the number of non-zero entries in $M$. 

Proof of Birkhoff’s and von Neumann’s theorem
The proofs we’ve done so far

True → Ford-Fulkerson

König-Egerváry

Menger

Hall ← König

Birkhoff - v. Neumann

Dilworth
The proofs we’ll still do

True \rightarrow Ford-Fulkerson

Berge

König-Egerváry

Hall

König

Birkhoff - v.Neumann

Menger

Dilworth
A direct proof for Hall’s theorem

- Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall’s criterion.
- If $\forall x \in X : \deg(x) = 1$, then the matching is obvious.
- Let $x \in X$ be such that $\deg(x) \geq 2$. Let $(x, y_1), (x, y_2) \in E$.
- Assume that we can remove neither $(x, y_1)$ nor $(x, y_2)$ without violating Hall’s criterion.
- There are $S_1, S_2 \subseteq X \setminus \{x\}$, such that
  \[
  |N(S_i) \cup (N(x) \setminus \{y_i\})| < |S_i| + 1.
  \]

- Hence we get a contradiction:
  \[
  |S_1| + |S_2| \geq |N(S_1) \cup (N(x) \setminus \{y_1\})| + |N(S_2) \cup (N(x) \setminus \{y_2\})| \\
  \geq |N(S_1) \cup (N(x) \setminus \{y_1\}) \cup N(S_2) \cup (N(x) \setminus \{y_2\})| + |N(S_1) \cap N(S_2)| \\
  \geq |N(S_1 \cup S_2 \cup \{x\})| + |N(S_1 \cap S_2)| \geq |S_1 \cup S_2| + 1 + |S_1 \cap S_2| = |S_1| + |S_2| + 1
  \]
Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall’s criterion.
Let $P = (X, cupY, \leq)$ be a partially ordered set:
- $x < y$ iff $x \in X$, $y \in Y$ and $(x, y) \in E$

$Y$ is an antichain in $P$.
If $Z$ is any antichain in $P$, then $N(Z \cap X) \cap (Z \cap Y) = \emptyset$. Hence

$$|Z| = |Z \cap X| + |Z \cap Y| \leq |N(Z \cap X)| + |Z \cap Y| \leq |Y| .$$

$P$ can be partitioned to $|Y|$ chains.
Each element of $X$ will be in a chain together with an element of $Y$.
These give us the matching.
Let $G = (V, E)$ be a graph and $M \subseteq E$ a matching in it.

**Definition**

- An open path $P$ in $G$ is **$M$-alternating ($M$-vahelduv)** if the edges of $P$ alternately belong to $M$ and $E\setminus M$.
- An alternating path $P$ with endpoints $x$ and $y$ is **$M$-augmenting ($M$-laienev)** if $\deg_M(x) = \deg_M(y) = 0$.

**Theorem (Berge)**

A matching $M$ in graph $G$ is maximal iff there are no $M$-augmenting paths in $G$. 
Proof of Berge’s theorem

$(\Rightarrow)$: If $P$ is an $M$-augmenting path then $M' = (M \setminus P) \cup (P \setminus M)$ is a matching and $|M'| = |M| + 1$.

$(\Leftarrow)$: Let $M$ be a non-maximal matching in $G$. Let $M^*$ be a matching with $|M^*| > |M|$.

Consider the graph $H = (V, M \cup M^*)$.

- $\forall v \in V: \deg_H(v) \geq 2$.

There are following kinds of connected components in $H$:

- Isolated vertices.
- Cycles (of even length).
- Two vertices connected by an edge from $M \cap M^*$.
- Paths, where edges from $M$ and $M^*$ alternate.

There must be a connected component having more edges from $M^*$ than from $M$.

Only possibility: path of odd length, starting and ending with an edge from $M^*$.

This is an $M$-augmenting path.
Let $G = (X \cup Y, E)$ be a bipartite graph. Let $M$ be a maximum
matching in it. Let $x \in X$ be uncovered by $M$.

Construct all possible $M$-alterating paths starting from $x$.

Let $S \subset X$ be the set of vertices in $X$ on these paths (incl. $x$).

Let $T \subset Y$ be the set of vertices in $Y$ on these paths.

We have

- $N(S) = T$, because any edge from some $u \in S$ can continue an
  $M$-alternating path.
- $|S\{x}\| = |T|$. The edges in $M$ give a bijection between $S\{x\}$ and $T$.
  - The non-existence of $M$-augmenting paths implies that any $M$-alternating
    path ending in $Y$ can be continued.

Hence $G$ does not satisfy Hall’s criterion.
Let $M$ be a $m \times n$ 0/1-matrix. Let its minimal cover consist of rows with indices in $R$ and columns with indices in $C$.

Let $G_R = (R \cup \overline{C}, E_R)$, where $E_R = \{(r, c) | M_{rc} = 1\}$.

Let $S \subseteq R$. Then $|N(S)| \geq S$, as otherwise the rows in $S$ could be replaced with the smaller number of columns in $N(S)$, still covering all 1-s in $M$.

Thus $M$ has a partial transversal in rows $R$ and columns outside of $C$, such that its size is $|R|$.

Similarly, let $G_C = (C \cup \overline{R}, E_C)$, where $E_C = \{(c, r) | M_{rc} = 1\}$.

There’s a partial transversal in columns $C$ and rows outside of $R$, such that its size is $|C|$.

Joining these partial transversals, we get a partial transversal of size $|R| + |C|$.