Discrete Mathematics, 3rd lecture

Eulerian and Hamiltonian graphs

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Recap: graphs

- [Undirected] graph — triple $(V, E, \mathcal{E})$, where V — set of vertices, E — set of edges, $\mathcal{E}$ — the incidence function.
- Walk in the graph is a sequence

$$v_0 \overset{e_1}{—} v_1 \overset{e_2}{—} v_2 \overset{e_3}{—} v_3 \overset{e_4}{—} \ldots v_{k-1} \overset{e_k}{—} v_k$$

  where $v_0, \ldots, v_k \in V$, $e_1, \ldots, e_k \in E$ and $\mathcal{E}(e_i) = \{v_{i-1}, v_i\}$.
- A walk is closed if its first and last vertex coincide.
- A path is a walk where the vertices do not repeat.
- A cycle is a closed path.

Eulerian walks

Definition

- An Eulerian walk in a graph G is a closed walk that contains each edge exactly once.
- An Eulerian graph is a connected graph that contains an Eulerian walk.

Demanding connectedness removes some cases that do not add anything interesting, but just get in the way.

Definition

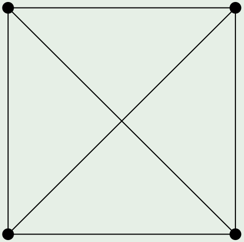
A semi-Eulerian graph is a graph that has an open walk that contains each edge exactly once.

A well-known class of puzzles

Draw the given figure without raising the pen from the paper and without repeating a line.

Examples

Example



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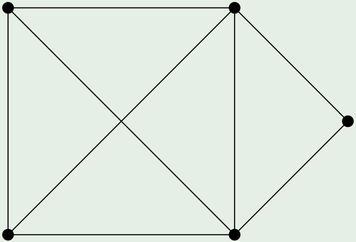

Examples

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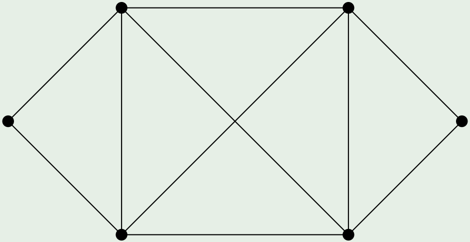
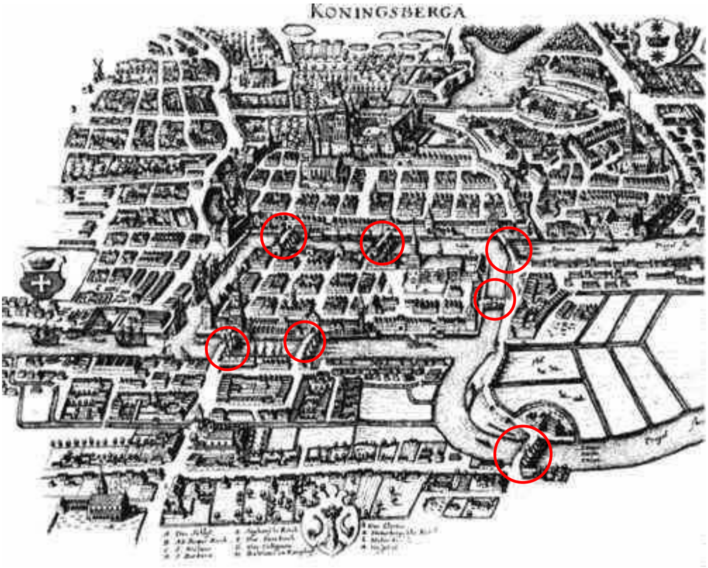
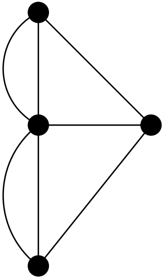


The “original task”





The necessary and sufficient condition

Theorem

Let $G = (V, E, \mathcal{E})$ be a connected graph. The following are equivalent

1. *G is Eulerian;*
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Theorem

Let $G = (V, E, \mathcal{E})$ be a connected graph. The following are equivalent

1. *G is Eulerian;*
2. *All vertices of G have even degree;*
3. *E can be partitioned into cycles.*

"Partitioned into cycles"

There are $E_1, \ldots, E_k \subseteq E$, such that

- $E_i \cap E_j = \emptyset$, if $i \neq j$;
- $E_1 \cup \cdots \cup E_k = E$;
- For each i, there is a cycle C_i in G, such that the edges of C_i are precisely the elements of E_i.

Proof (1) ⇒ (2)

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- $\deg(v) = \overrightarrow{\deg_P}(v) + \overleftarrow{\deg_P}(v)$ and $\overrightarrow{\deg_P}(v) = \overleftarrow{\deg_P}(v)$.
- Hence $\deg(v) = 2\overrightarrow{\deg_P}(v)$ is an even number.

Proof (2) $\Rightarrow$ (3)

Ideas?

Given: degrees of all vertices are even.
Also given: the graph is connected.

- We need to partition E into cycles.

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- And where do we actually get that cycle from?

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- How about the assumption (2)? Will it still hold if we remove C?
- How do the degrees of of vertices change if we remove a cycle?
- And where do we actually get that cycle from?
    - Remember something from the last lecture?

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If $E = \emptyset$ then E can be partitioned to zero parts, each of which...

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- Remove [edges of] C from G. All vertices still have even degree.

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- Remove [edges of] C from G. All vertices still have even degree.
- Let $C_1 \dot\cup \cdots \dot\cup C_k$ be a partition of $E \backslash C$ into cycles.
  - Exists by the induction hypothesis.

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Exercise. Where did we use the connectedness of G?

Proof (3) $\Rightarrow$ (1)

Ideas?

Given: the edges of a connected graph $G = (V, E, \mathcal{E})$ are partitioned to cycles.

- We have cycles $C_1, \ldots, C_k$ without common edges. We have to pass through all of them.

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- Looks like induction again.
- Where is connectedness important?

Proof (3) $\Rightarrow$ (1)

Using connectedness

$E = C_1 \dot{\cup} \cdots \dot{\cup} C_k$. We may assume w.l.o.g. that $\forall i \in \{2, \ldots, k\}. \exists j \in \{1, \ldots, i-1\}$, such that C_i and C_j have a common vertex.

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E is a single cycle. Eulerian walk goes through it.

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Induction basis

E is a single cycle. Eulerian walk goes through it.

Induction step

- Let P be a closed walk passing through edges in $C_1 \cup \cdots \cup C_{k-1}$.
    - Exists because of the induction hypothesis.
- P passes through some vertex on C_k.
- At this vertex, interrupt P, pass around C_k, continue with P.

□

Algorithm for finding an Eulerian walk

Implicit in the proof of the previous theorem.

1. Partition the edges of the graph into cycles.
2. Pass through all of them.

Semi-Eulerian graphs

Theorem

A connected graph is semi-Eulerian iff it has exactly two vertices with odd degree.

Exercise. Prove it, by using the previous theorem in a material way.

Hamiltonian graphs

Definition

- Hamiltonian cycle in graph G is a cycle that passes through each vertex exactly once.
- Hamiltonian path in graph G is an open path that passes through each vertex exactly once.
- If a graph has a Hamiltonian cycle, it is called a Hamiltonian graph.
- If a graph has a Hamiltonian path (but no cycle), it is called a semi-Hamiltonian graph.

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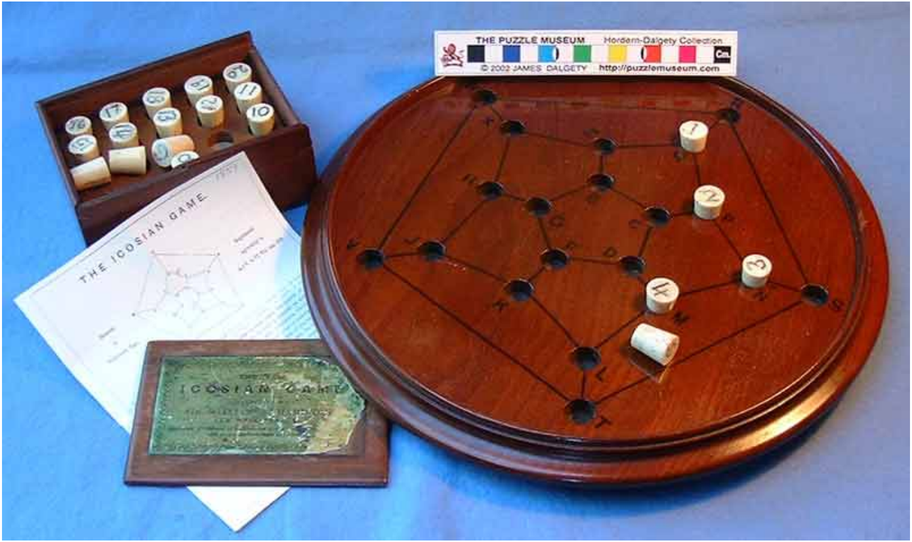
- In Hamiltonicity considerations, double edges and loops are irrelevant.
- Hence we only consider simple graphs $G = (V, E)$.

Sir William Rowan Hamilton's Icosian game



Necessary and sufficient conditions for Hamiltonicity

- For Eulericity, there was a nice, locally checkable necessary and sufficient condition.
- No such condition is known for Hamiltonicity.
- The question, whether a given graph G is Hamiltonian or not, is NP-complete.
  - Hence the existence of a simple algorithm for checking it is unlikely.
- There exist easily checkable, sufficient, but **not** necessary conditions for Hamiltonicity.
  - Many of them are are variations of “if a graph has many edges then it is Hamiltonian”.

If a graph has many edges. . .

Theorem (Dirac, 1952)

If a simple graph $G = (V, E)$ with $|V| = n \geqslant 3$ satisfies

$$\forall v \in V : \deg(v) \geqslant n/2$$

then G is Hamiltonian.

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Theorem (Dirac, 1952)

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then G is Hamiltonian.

Follows trivially from

Theorem (Ore, 1960)

If a simple graph $G = (V, E)$ with $|V| = n \geqslant 3$ satisfies

$$\forall u, w \in V : \Big(\text{if } (u, w) \notin E \text{ then } \deg(u) + \deg(w) \geqslant n\Big) \qquad \text{(O)}$$

then G is Hamiltonian.

Proof of Ore theorem

$n = 3$

If $n = 3$ then the only graph satisfying (O) is K_3. It is Hamiltonian.

Exercise. Verify that only K_3 satisfies (O).

Let $n \geqslant 4$.

Do proof by contradiction

Assume there exists a non-Hamiltonian graph G satisfying (O).

Lemma

If $G = (V, E)$ satisfies (O) and $(u, v) \notin E$, then $G' = (V, E \cup \{(u, v)\})$ also satisfies (O).

Exercise. Prove it.

Lemma (The limit graph G^*)

There exists a graph $G^* = (V, E^*)$ satisfying (O) and $E \subseteq E^*$, such that

- G^* is not Hamiltonian.
- The addition of any one edge would make G^* Hamiltonian.

Why can and should we go to the limit?

Proof of the lemma

Add arbitrary edges to G until you reach that G^*.

- By adding edges to G, we will eventually obtain a Hamiltonian graph.
  - Because K_n is Hamiltonian.
  - And we can only add a finite number of edges.
- We stop one step before obtaining a Hamiltonian graph
  - …when there is no way to add one more edge.

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  - ... when there is no way to add one more edge.

- We know more about G^* than we know about G.
- So we can argue more about it.
- Note that $G^* \neq K_n$.

Proof of Ore theorem

G^* is semi-Hamiltonian

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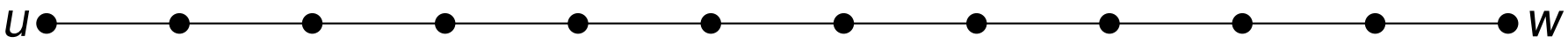
- $G^* \cup \{e\}$ is Hamiltonian. Let C be a Hamiltonian cycle in $G^* \cup \{e\}$.
- C uses the edge e.
- $C \backslash \{e\}$ is a path starting in u and ending in w and going through all vertices in V.
  - It uses only edges in E^*.
  - Its length is $n - 1$.

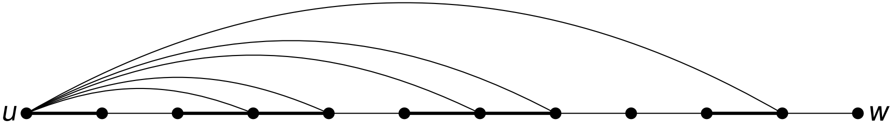
- The Hamiltonian path.



Proof of Ore theorem

G^* is actually Hamiltonian

- The Hamiltonian path.
- How many edges have their right end-point connected to u?



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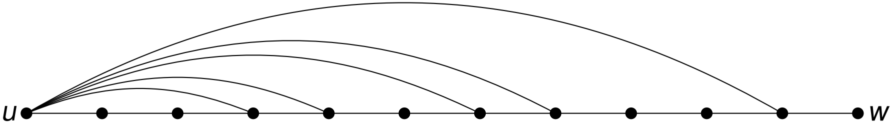

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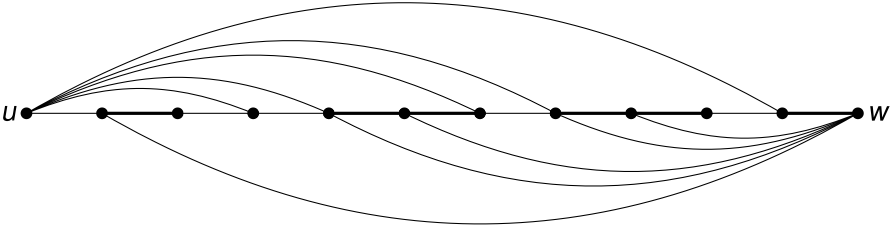

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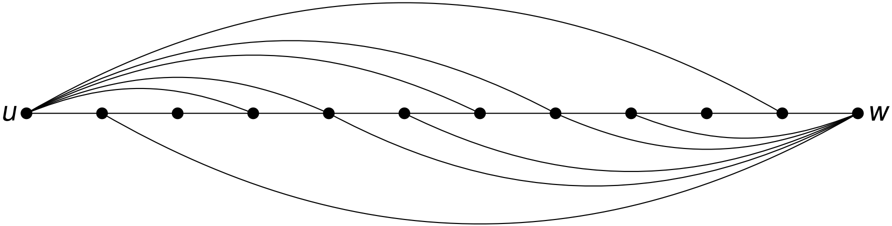

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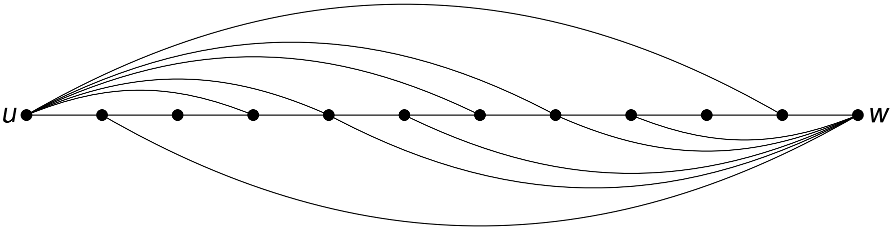
- There are $n-1$ edges on the path. $\deg(u) + \deg(w) \geqslant n$.



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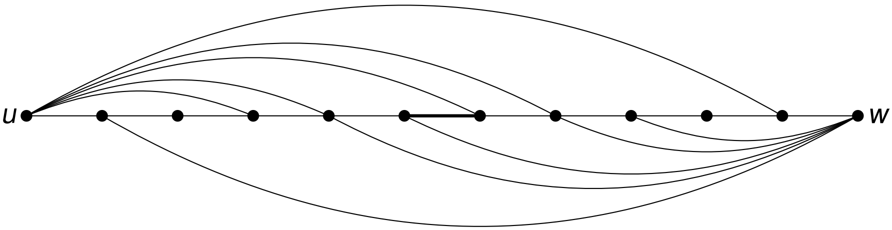

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- There are $n - 1$ edges on the path. $\deg(u) + \deg(w) \geqslant n$.
- Some edge has both end-points connected.
- These edges give us a Hamiltonian cycle. □