Graphs

(MTAT.05.080, 4 AP / 6 ECTS)

Lectures:	Fri 12-14, hall 405
Exercises:	Mon 14-16, hall 315
või	N 12-14, aud. 405

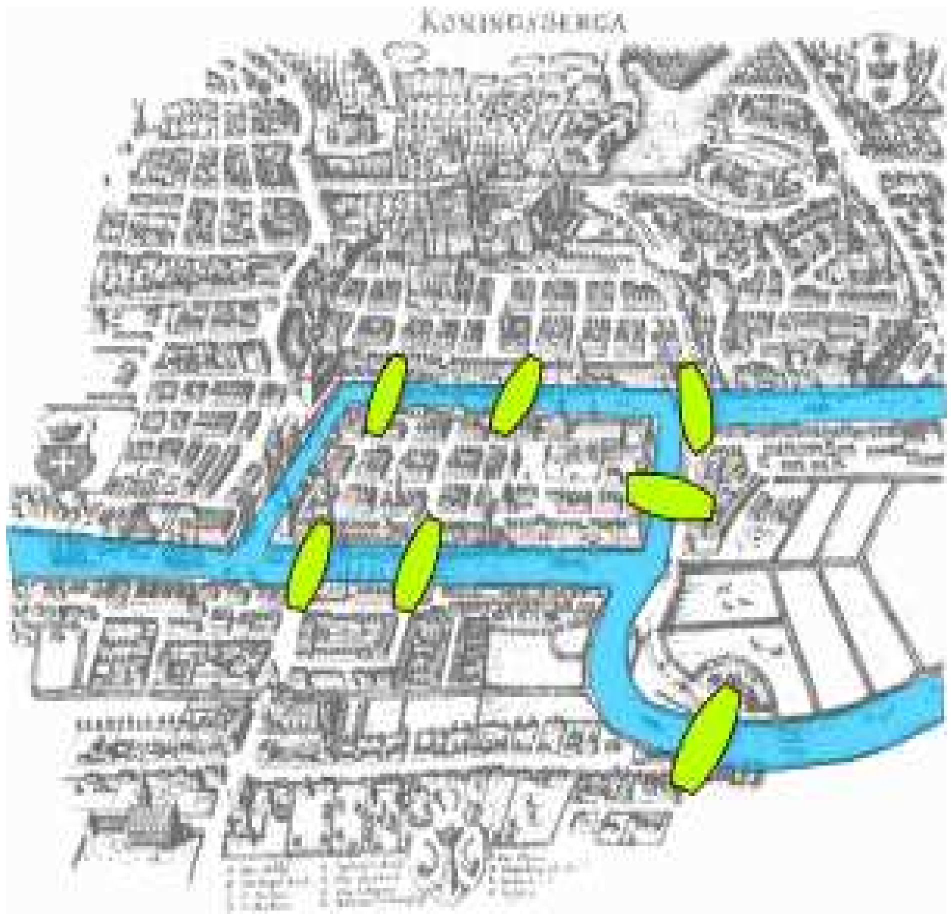
homepage:

`http://www.ut.ee/~peeter_l/teaching/graafid08s`

(contains slides)

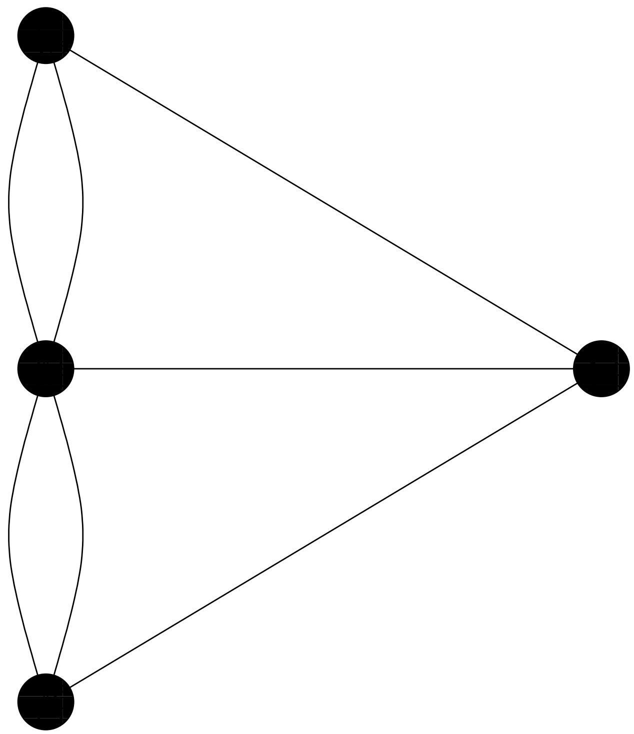
For grade: three tests (during or after semester)

Königsberg, 1736



Does there exist a walk that would cross every bridge exactly once and return to the beginning?

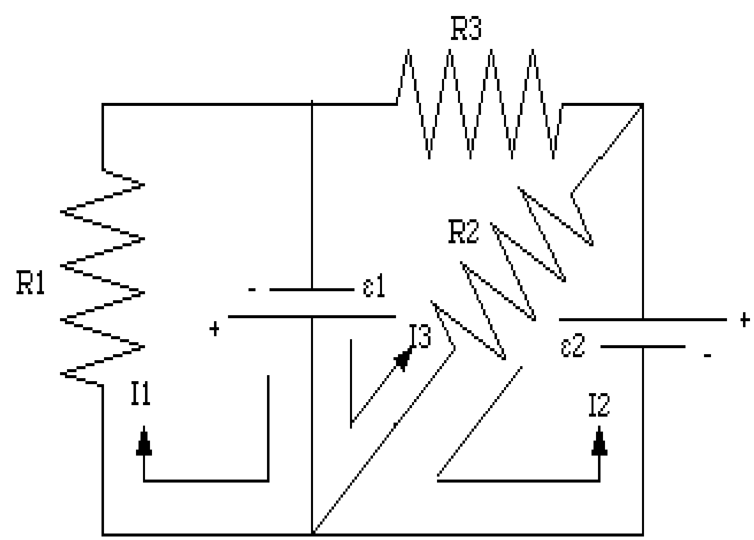
Graphical representation of the Königsbergi bridge problem:



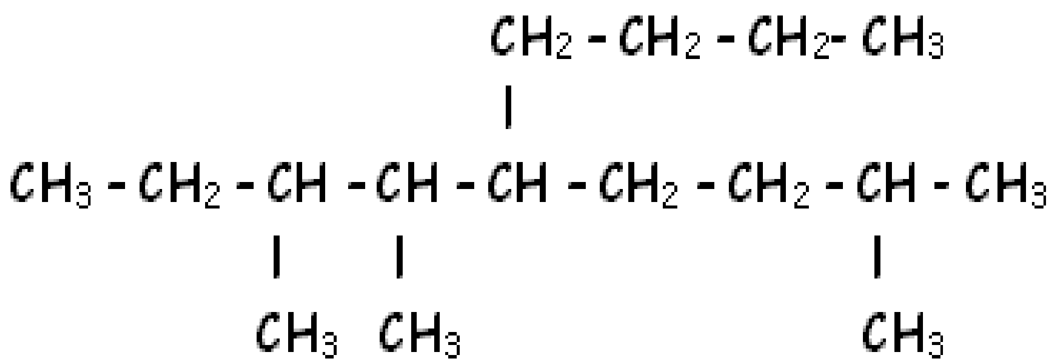
Euler: “Such a walk is impossible!”

Kirchhoff laws (1847):

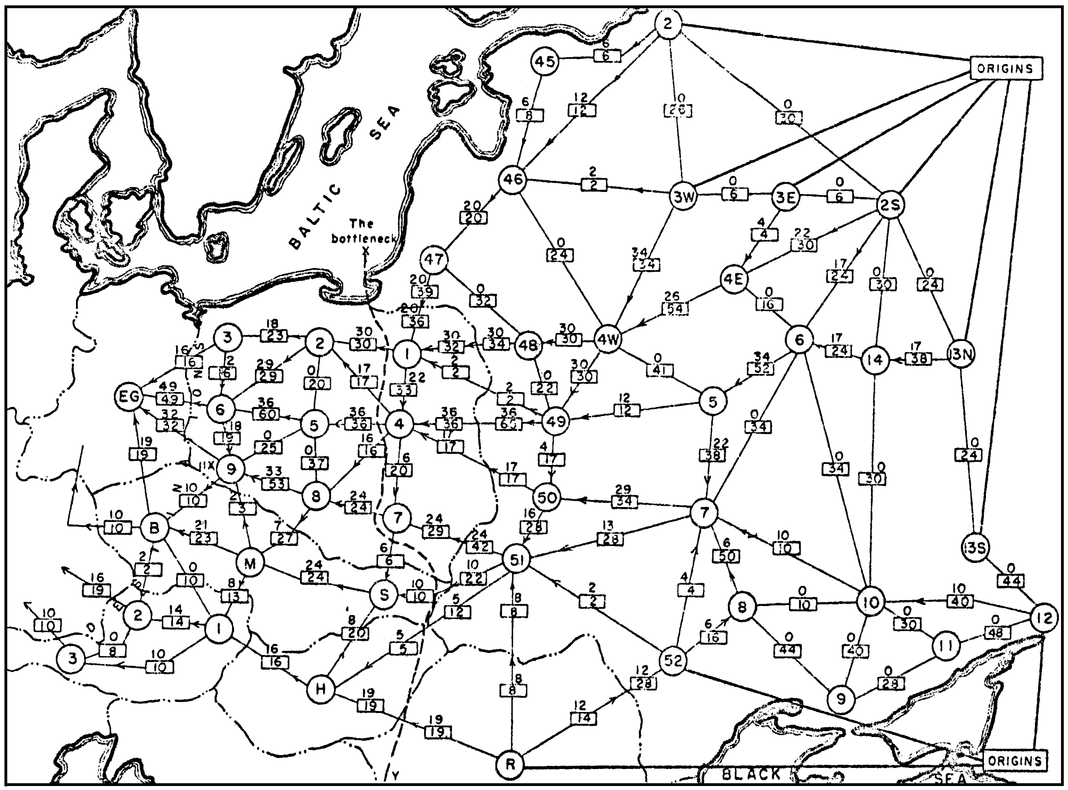
- At any point in an electrical circuit, the sum of currents flowing towards that point is equal to the sum of currents flowing away from that point.
- The directed sum of the electrical potential differences around a circuit must be zero.



Sir Arthur Cayley (1857): “How many isomers of alcanes C_nH_{2n+2} are there?”



USA army (1955): “How fast is it possible to transport supplies from Soviet Union to Eastern Europe?”



Ford&Fulkerson: “Using the railway system, up to 163000 tons as a time.”

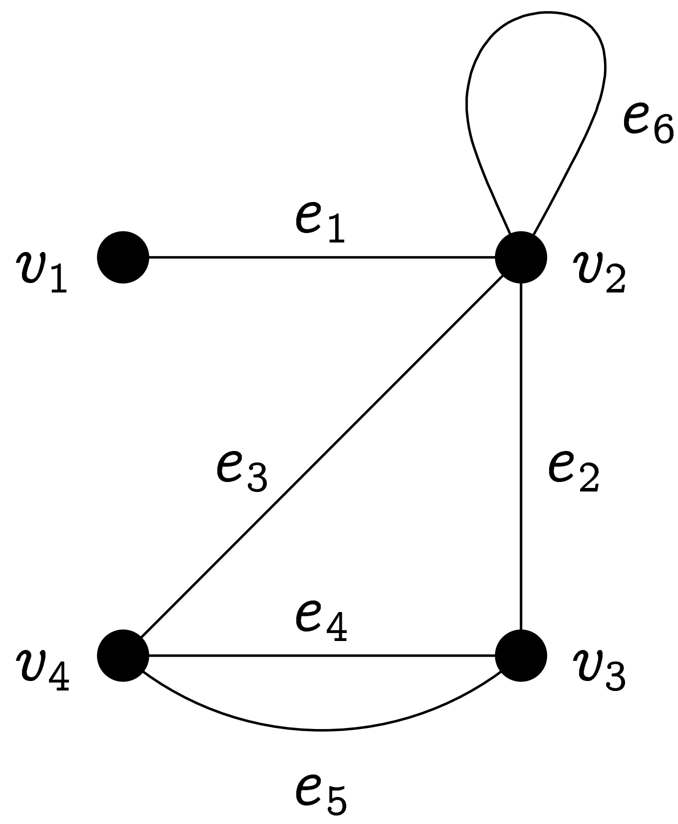
(Undirected) graph is a triple $G = (V, E, \mathcal{E})$, where

- V is the set of *vertices* (also denote $V(G)$);
- E is the set of *edges* (also denote $E(G)$).
- $\mathcal{E} : E \longrightarrow \mathcal{P}(V)$ is the *incidency mapping*. For all $e \in E$, $\mathcal{E}(e)$ must have 1 or 2 elements.

In this course we only consider finite graphs.

Example: Let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and

e	$\mathcal{E}(e)$
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_4\}$
e_4	$\{v_3, v_4\}$
e_5	$\{v_3, v_4\}$
e_6	$\{v_2\}$



A drawing may <u>illustrate</u> a graph.

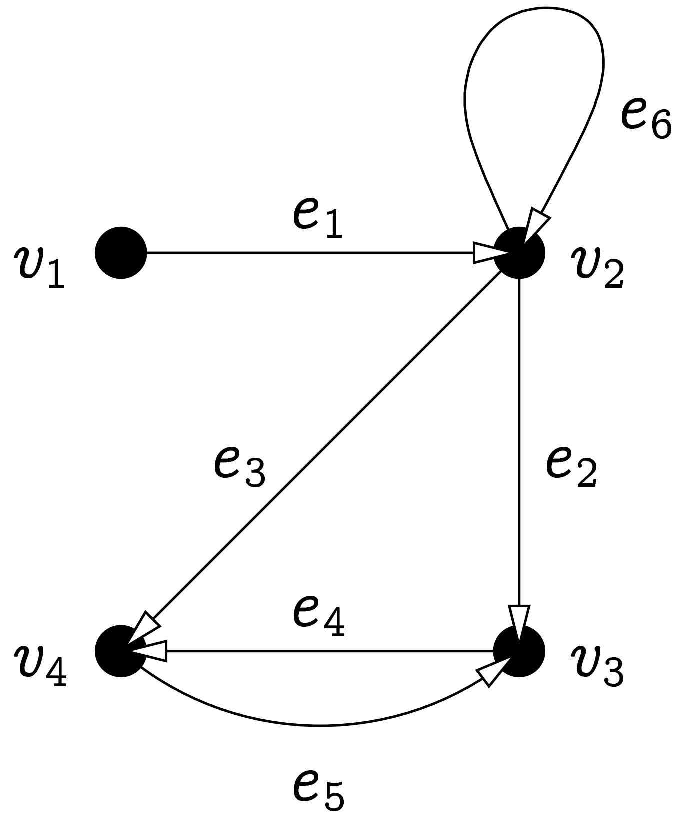
But a graph itself is still the triple $(V, E, \mathcal{E})$.

Directed graph consists of the sets of vertices V and edges E, and the incidency mapping $\mathcal{E} : E \longrightarrow V \times V$.

The edges of a directed graph may also be called *arcs*.

Example: let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and

e	$\mathcal{E}(e)$
e_1	(v_1, v_2)
e_2	(v_2, v_3)
e_3	(v_2, v_4)
e_4	(v_3, v_4)
e_5	(v_4, v_3)
e_6	(v_2, v_2)



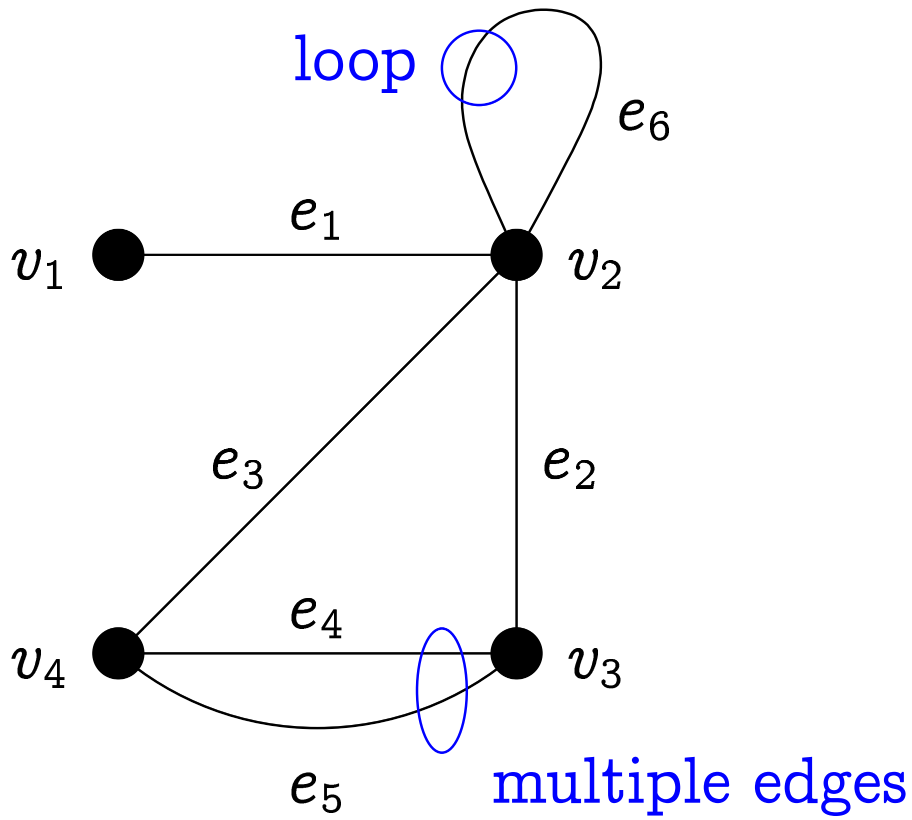
Let $G = (V, E, \mathcal{E})$ be a graph. Notations:

- If $v \in \mathcal{E}(e)$, then v and e are *incident*.
- If there exists e, such that $\mathcal{E}(e) = \{v_1, v_2\}$, then v_1 and v_2 are *adjacent*.
- If $\mathcal{E}(e) = \{v_1, v_2\}$, then v_1 and v_2 are the *endpoints* of e. Denote also $v_1 \overset{e}{\text{—}} v_2$.

Let $G = (V, E, \mathcal{E})$ be a directed graph. Notations:

- If $\mathcal{E}(e) = (v_1, v_2)$, then v_1 is the *start vertex* and v_2 the *end vertex* of e.

$e \in E$ is a *multiple edge*, if there exists $e' \in E \setminus \{e\}$, such that $\mathcal{E}(e) = \mathcal{E}(e')$. $e \in E$ is a *loop*, if $|\mathcal{E}(e)| = 1$.



A *simple graph* is a graph without loops and multiple edges.

In a directed simple graph, we may assume $E \subseteq V \times V$. To an edge $e \in E$, where $\mathcal{E}(e) = (v_1, v_2)$, corresponds $(v_1, v_2) \in V \times V$.

Example: let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$ and

e	$\mathcal{E}(e)$
e_1	(v_1, v_2)
e_2	(v_2, v_3)
e_3	(v_2, v_4)
e_4	(v_3, v_4)

We may assume that $E = \{(v_1, v_2), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$.

In an undirected simple graph we may also assume $E \subseteq V \times V$.

To an edge $e \in E$, where $\mathcal{E}(e) = \{v_1, v_2\}$, corresponds $\{(v_1, v_2), (v_2, v_1)\} \subseteq V \times V$.

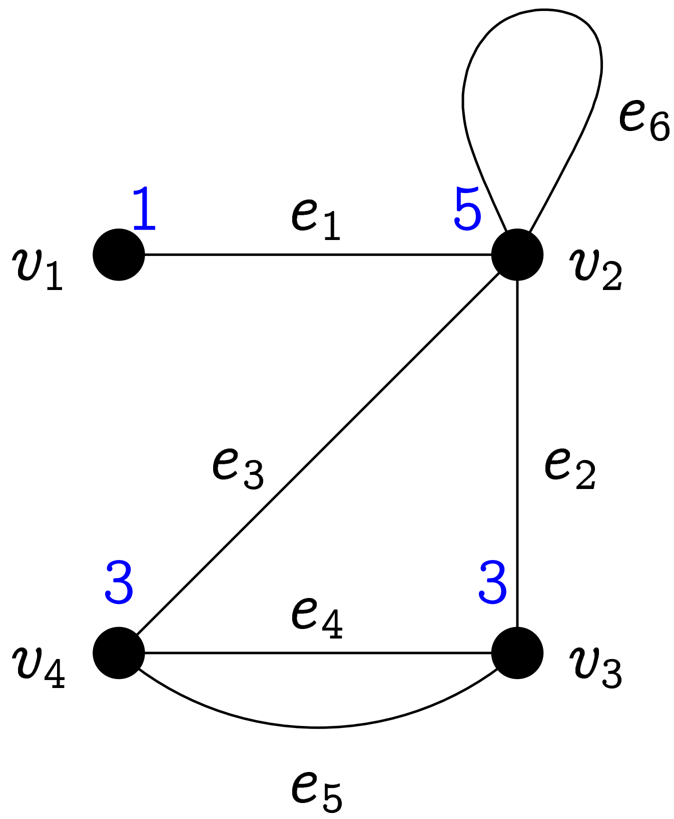
Example: let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$ and

e	$\mathcal{E}(e)$
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_4\}$
e_4	$\{v_3, v_4\}$

We may assume that $E = \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2), (v_2, v_4), (v_4, v_2), (v_3, v_4), (v_4, v_3)\}$.

The *degree* of a vertex v in the graph $(V, E, \mathcal{E})$ is the number of edges incident to it (the loops count twice). Denote $\deg(v)$.

$$\deg(v) = |\{e \in E \mid v \in \mathcal{E}(e)\}| + |\{e \in E \mid \mathcal{E}(e) = \{v\}\}|$$

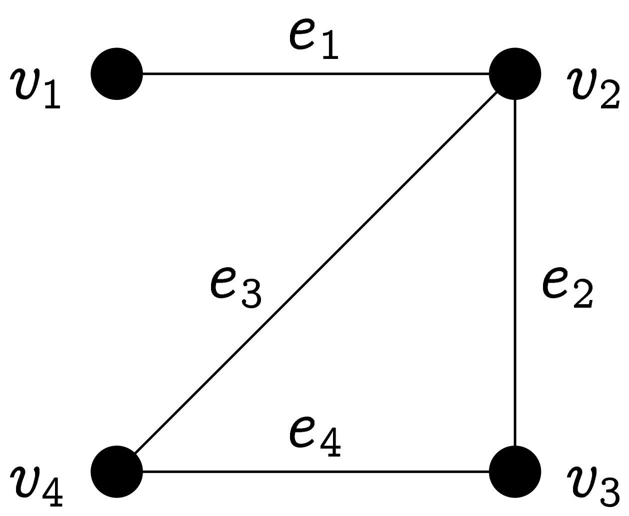


Let $G = (V, E)$ be undirected simple graph. Let $V = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, where

- If $(v_i, v_j) \in E$, then $a_{ij} = 1$.
- If $(v_i, v_j) \notin E$, then $a_{ij} = 0$.

The adjacency matrix is symmetric and its main diagonal contains zeroes.

Example



	1	2	3	4
1	0	1	0	0
2	1	0	1	1
3	0	1	0	1
4	0	1	1	0

Theorem. An undirected simple graph contains an even number of vertices of odd degree.

Proof. Count the ones in the adjacency matrix of $G = (V, E)$.

- Their number is $2 \cdot |E|$.
- Their number is $\sum_{v \in V} \deg(v)$.

These two quantities are equal. Hence the sum of degrees of all vertices is even. The sum of integers is even if an even number of summands are odd. $\square$

Similarly, any undirected graph contains an even number of vertices of odd degree.

Proof. Let $V = \{v_1, \ldots, v_n\}$. Consider the sets

$$\begin{array}{ll}
\{e \in E \mid v_1 \in \mathcal{E}(e)\} & \{e \in E \mid \mathcal{E}(e) = \{v_1\}\} \\
\{e \in E \mid v_2 \in \mathcal{E}(e)\} & \{e \in E \mid \mathcal{E}(e) = \{v_2\}\} \\
\multicolumn{2}{c}{\dotfill} \\
\{e \in E \mid v_n \in \mathcal{E}(e)\} & \{e \in E \mid \mathcal{E}(e) = \{v_n\}\}
\end{array}$$

Each $e \in E$ belongs to exactly two of those sets.

Hence $\sum_{v \in V} \deg(v) = 2 \cdot |E|$, which is even. $\square$

In a directed graph $(V, E, \mathcal{E})$ we define for a vertex v

- its *indegree* $\overrightarrow{\deg}(v)$ — the number of edges ending in v;
- *outdegree* $\overleftarrow{\deg}(v)$ — the number of edges starting in v.

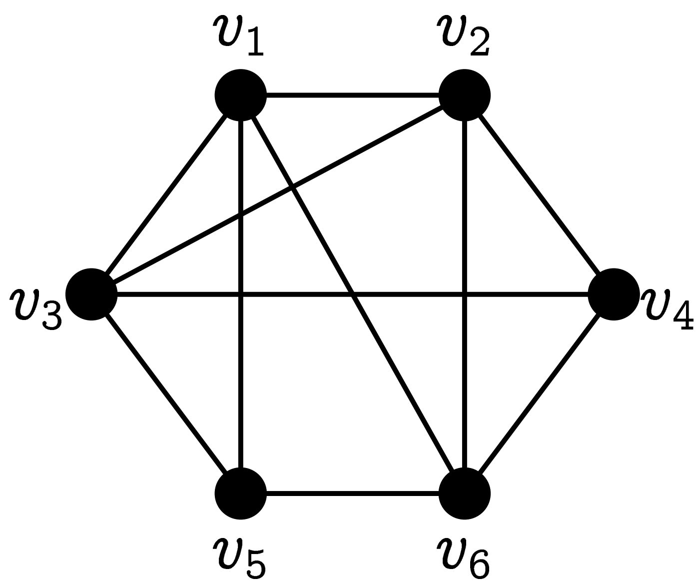
Similar theorem: $\sum\limits_{v \in V} \overrightarrow{\deg}(v) = \sum\limits_{v \in V} \overleftarrow{\deg}(v)$.

- A *walk* in the graph $G = (V, E)$ (from vertex x to vertex y) is a sequence

$$P : x = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} v_3 \xrightarrow{e_4} \dots v_{k-1} \xrightarrow{e_k} v_k = y \quad .$$

- The number k is the *length* of the walk P. Denote $|P|$.
- Let $x \overset{P}{\rightsquigarrow} y$ denote that P is a walk from x to y.
- A *path* is a walk where all vertices are distinct (only v_0 and v_k may coincide).
- A walk is *closed* if $v_0 = v_k$.
- A closed path is a *cycle*.
- A graph is *connected* if there is a walk between each two of its vertices.
- The *distance* $d(u, v)$ between vertices $u, v \in V$ is the length of the shortest walk connecting them.

Examples:



Walk: $v_1 — v_2 — v_4 — v_6 — v_2 — v_3$

Path: $v_1 — v_2 — v_3 — v_4$

Closed walk: $v_1 — v_2 — v_3 — v_1 — v_5 — v_6 — v_1$

Cycle: $v_1 — v_2 — v_6 — v_5 — v_1$

$d(v_1, v_4) = 2$, $d(v_1, v_2) = 1$, $d(v_1, v_1) = 0$.

Theorem. If the degree of each vertex of a graph is at least 2, then this graph contains a cycle.

Proof. Loops and multiple edges are cycles.

Assume $G = (V, E)$ is simple. Let $v_1 \in V$. Exists $v_2 \in V$, such that $v_1 — v_2$. Exists $v_3 \in V$, such that $v_1 — v_2 — v_3$. This path is simple.

Let $v_1 — v_2 — \cdots — v_k$ be a simple path. There exists $v_{k+1} \in V$, such that $v_{k+1} \neq v_{k-1}$ and $v_k — v_{k+1}$.

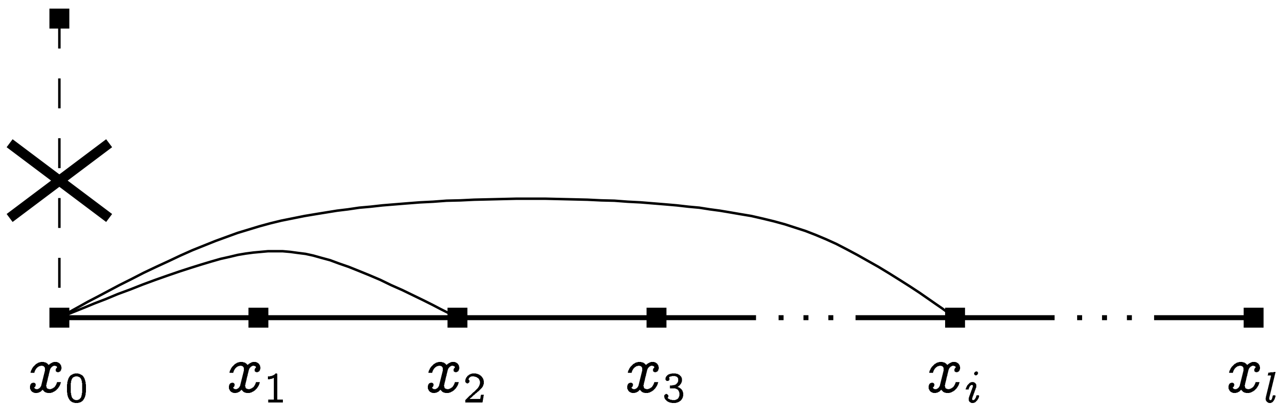
If $v_{k+1} = v_i$ for some $i \in \{1, \ldots, k-2\}$, then we have a cycle.

Otherwise we have a longer simple path $v_1 — v_2 — \cdots — v_k — v_{k+1}$.

The length of a simple path is bounded by $|V|$. $\square$

Theorem. A simple graph, where the degree of each vertex is at least $k \geqslant 2$, has a cycle of length at least $k+1$.

Proof. Let $x_0 — x_1 — \cdots — x_l$ be an open path of maximal length in this graph.



All neighbours of x_0 are located in this path.

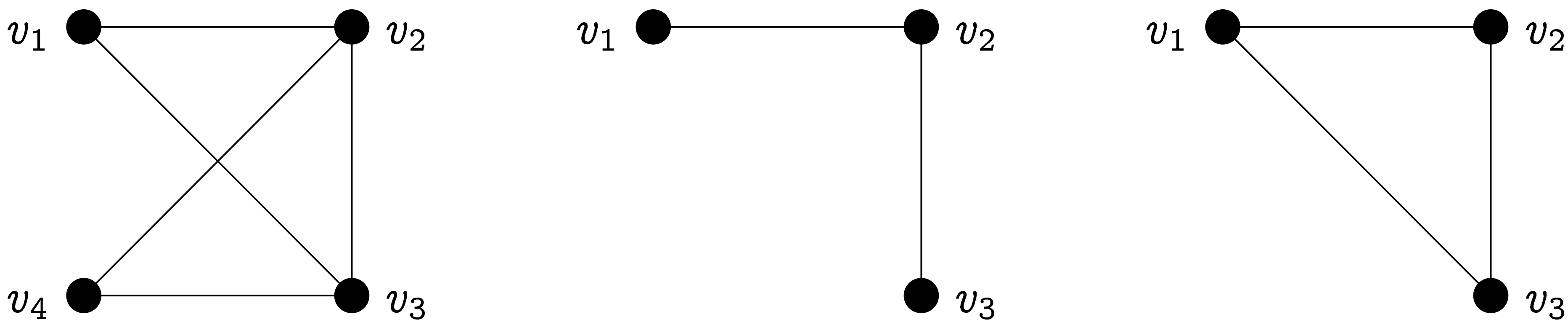
Let x_i be the neighbour of x_0 with maximal index. Then $i \geqslant k$.

$x_0 — x_1 — \cdots — x_i — x_0$ is a cycle of length $i+1 \geqslant k+1$. $\square$

The *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$, where $V' \subseteq V$, $E' \subseteq E$ and for all $e \in E'$ holds $\mathcal{E}(e) \subseteq V'$. Denote $G' \leqslant G$.

A subgraph (V', E') is *induced* (by the set V'), if the set E' is as large as possible, i.e. $\mathcal{E}(e) \subseteq V' \Rightarrow e \in E'$ holds for all $e \in E$.

Example:



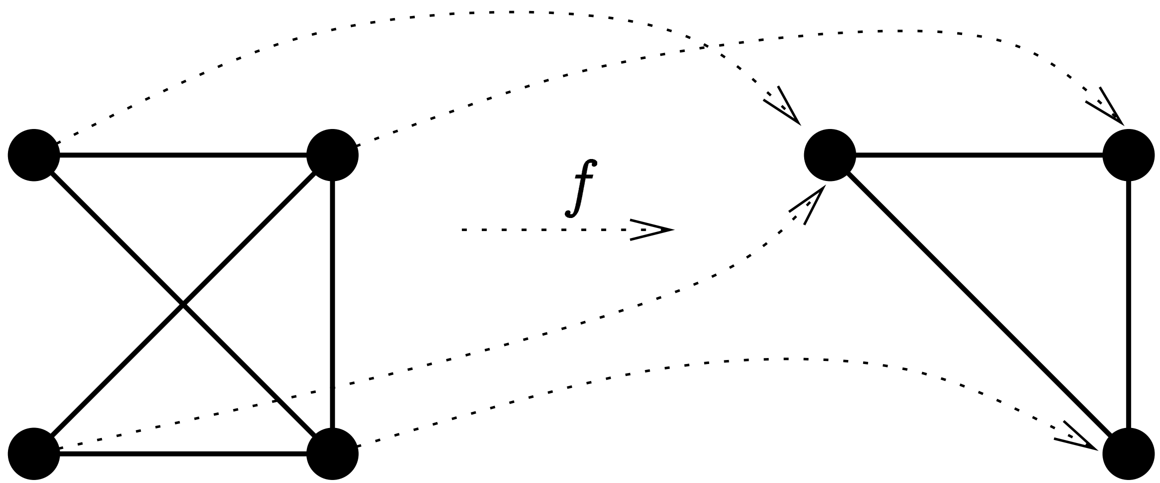
The *connected components* of a graph G are its maximal connected subgraphs.

More notions:

- An edge of a graph is *bridge* if its removal increases the number of connected components.
- A vertex of a graph is *cut vertex* if its removal (together with its incident edges) increases the number of connected components.

A *homomorphism* from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$ is a mapping $f : V_1 \longrightarrow V_2$, such that $x, y \in V_1$ are adjacent iff $f(x), f(y) \in V_2$ are adjacent.

Example:



Homomorphism f is *monomorphism*, if it's injective.

Homomorphism f is *isomorphism*, if it's bijective.

Graphs G_1 and G_2 are *isomorphic* (denote $G_1 \cong G_2$), if there exists an isomorphism between them.

Usually we identify isomorphic graphs with each other.

For example, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that G_1 is a subgraph of G_2, if there exist $V_1' \subseteq V_2$ and $E_1' \subseteq E_2$, such that $G_1 \cong (V_1', E_1')$ and $(V_1', E_1') \leqslant G_2$.

G_1 is an induced subgraph of G_2 iff there exists a monomorphism from G_1 to G_2.

Graph isomorphism problem: Find whether two given simple graphs are isomorphic.

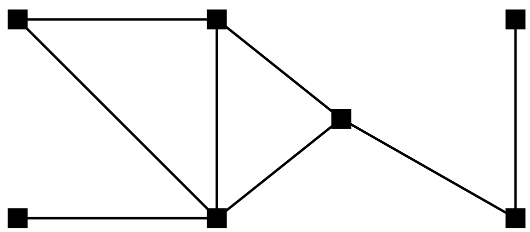
The graphs are represented by e.g. their adjacency matrices.

No polynomial-time algorithm for that task is known.

In the following, let us see some polynomial-time verifiable necessary conditions for the isomorphism of two graphs.

- Isomorphic graphs have equally many vertices and equally many edges.

The *degree sequence* of the graph $G = (\{v_1, \ldots, v_n\}, E)$ is the non-decreasing sequence of values $\deg(v_1), \ldots, \deg(v_n)$. For example, the degree sequence of the graph



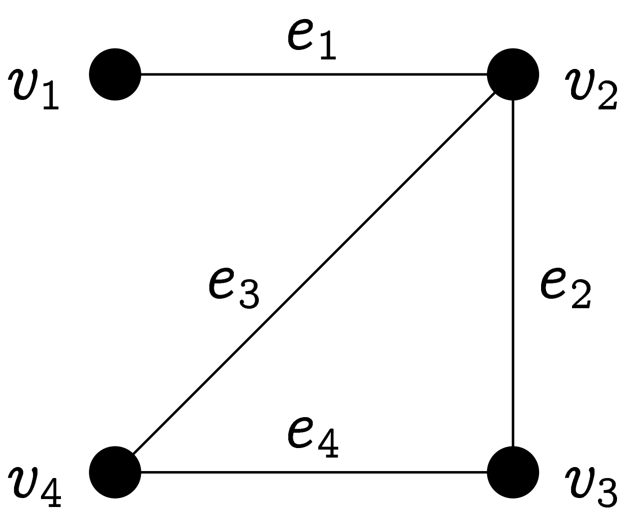
is $(1, 1, 2, 2, 3, 3, 4)$.

- Isomorphic graphs have the same degree sequences.

Let A be a square matrix and E a unit matrix of the same dimensions. The *characteristic polynomial* (of the variable x) of the matrix A is

$$\det(A - xE) \ .$$

The *characteristic polynomial* of a simple graph is the characteristic polynomial of its adjacency matrix. E.g.:



$$\det \begin{pmatrix} -x & 1 & 0 & 0 \\ 1 & -x & 1 & 1 \\ 0 & 1 & -x & 1 \\ 0 & 1 & 1 & -x \end{pmatrix}$$
$$= x^4 - 4x^2 - 2x + 1$$

Two simple graphs are isomorphic iff the first adjacency matrix can be transformed to the second by permuting its rows and columns in the same way.

Such permutation does not change the elements on the main diagonal.

Such permutation contains an even number of swaps of rows or columns, hence it does not change the determinant.

- Isomorphic graphs have the same characteristic polynomial.

• An *edgeless graph* is a graph without edges. A null graph of n vertices is denoted by O_n or N_n.

• A *complete graph* is a simple graph with an edge between each pair of vertices. A complete graph of n vertices is denoted by K_n.

Proposition. Graph K_n has $\frac{n(n-1)}{2}$ edges.

• Graph $G = (V, E)$ is *bipartite*, if V can be partitioned to two sets V_1 and V_2 (i.e. $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$), such that the endpoints of any edge belong in different parts.

(More generally: a graph is k-partite if its vertices can be partitioned into k parts such that all edges are between different parts.)

A bipartite simple graph with parts of vertices V_1 and V_2 is *complete bipartite* if there is an edge between each $v_1 \in V_1$ and $v_2 \in V_2$. Let $K_{m,n}$ denote the complete bipartite graph with $|V_1| = m$ and $|V_2| = n$.

Proposition. $K_{m,n}$ has mn edges.

Theorem. A graph is bipartite $\Leftrightarrow$ all its cycles are of even length.

Proof $\Rightarrow$. A cycle goes a number of times from the first part to the second and the same number of times from the second part to the first.

Proof $\Leftarrow$. Assume $G = (V, E)$ is connected. Otherwise consider each connected component separately.

In the following we'll colour the vertices of G black and white.

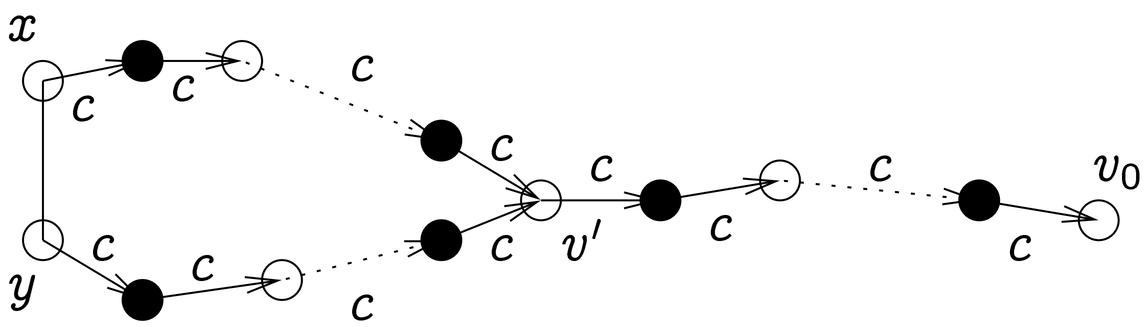
Pick a vertex $v_0 \in V$ and colour it white.

Let u be a coloured vertex that has uncoloured neighbours. Let v be one of such neighbours. Colour it with the opposite colour to u. Remember that the colour of u was used to choose the colour of v. Denote it $v \stackrel{c}{\to} u$.

Repeat the previous paragraph, until one of the following happens

- there appear adjacent vertices x and y of the same colour;
- we run out of vertices to colour.

If such vertices x and y appear then



we have a cycle of odd length $x - \cdots - v' - \cdots - y - x$.

Otherwise (we run out of vertices) the black vertices form one part and white vertices the other part of vertices of the bipartite graph. $\square$

Some history:

- 1735/6: Euler considered the Königsberg bridges question.
- 1750: Planar graphs: Euler's polyhedron formula. Proven 1794 by Legendre.
- *Knight's tour problem* (on chessboard): known hundreds of years. Solutions by Euler (1759), Vandermonde (1771).
- 1845: Kirchhoff's circuit laws.
- 1852: Earliest known mention of the four-color problem, in a letter from De Morgan to Hamilton.
- 1857: Icosian game (Hamilton).
- 1874: counting trees (Cayley). 1889: counting labeled trees.

- 1930: Necessary and sufficient criterions for planarity (Kuratowsky).
- 1976: Proof of the four-color theorem (Appel and Haken).

Möbius's puzzle to his students ($\approx$ 1840):

There once was a king with five sons. In his will he stated that after his death the sons should divide the kingdom into five regions, such that the boundary of each region should have a frontier line in common with each of the other four regions. Can the terms of the will be satisfied?

Tartu Akadeemiline Meeskoor võtab vastu uusi lauljaid.

Proovid toimuvad T 18:30 ja N 18:30 endises EPA klubis (Veski 6, Kassitoomel).

Uute lauljate vastuvõtt algab 7. septembrist.

Eriti oodatud on tenorid.