Planar graphs

A graph is *planar* (*tasandiline, planaarne*) if it can be drawn on a plane in such a way that its edges do not intersect outside their end-vertices.

Example: K_4 and Q_3 are planar, $K_{3,3}$ is not.

This definition is not precise because the notion of "drawing" is not precise.

The following illustrates the precise definition. However, in this lecture we will still use the intuitivity of the "definition" given above.

A *curve* (*kõver*) in the Euclidean space $\mathbb{R}^n$ is a function $\gamma : [a, b] \longrightarrow \mathbb{R}^n$, where $a, b \in \mathbb{R}$.

The curve γ is *continuous*, if $\lim_{x \to y} \gamma(x) = \gamma(y)$ for all $y \in [a, b]$.

The *length* of the curve γ is

$$\sup\{\sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)) \mid k \in \mathbb{N}, a = t_0 < t_1 < \ldots < t_k = b\} \ .$$

A curve is *rectifiable* (*sirgestuv*) if it has a length.

Let J_n be the set of all curves in the space $\mathbb{R}^n$ that are continuous, rectifiable, and do not intersect itself.

The *drawing* of a graph $G = (V, E)$ in the space $\mathbb{R}^n$ is a pair of mappings

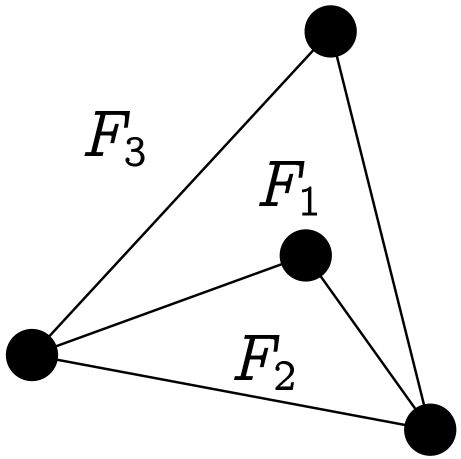
$$\iota_V : V \longrightarrow \mathbb{R}^n$$
$$\iota_E : E \longrightarrow J_n,$$

such that

- ι_V and ι_E are injective;
- if $\mathcal{E}(e) = \{u, v\}$, then the endpoints of $\iota_E(e)$ are $\iota_V(u)$ and $\iota_V(v)$.
- The curves $\iota_E(e_i)$ intersect each other only in their common end-points.

A graph is *planar* if it has a drawing in $\mathbb{R}^2$.

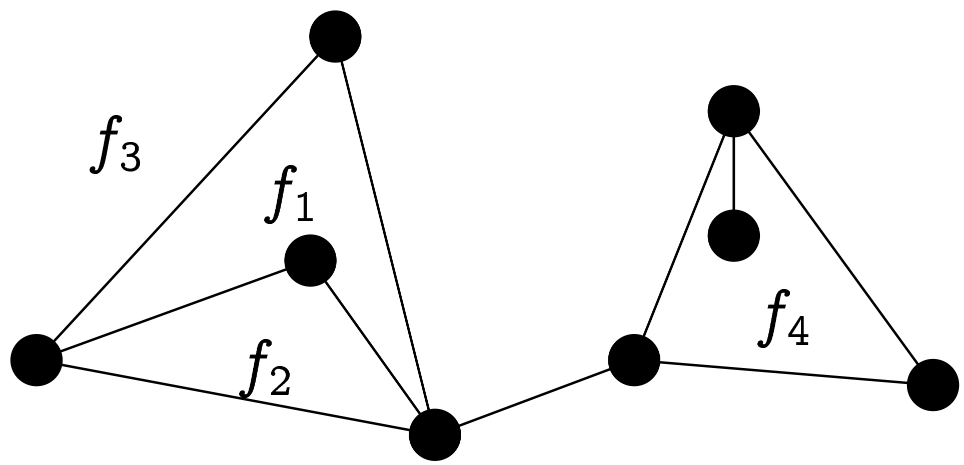
The drawing of the graph partitions the rest of the plane (not covered by the drawing) into *faces* (*tahk*). Graafi joonis tükeldab tasandi selle osa, mis joonise alla ei jää.



The face F_3 is the *infinite face*.

A graph can be drawn so, that any one of the faces was infinite.

$\Rightarrow$ A graph can be drawn so, that any one of the edges was outer.



Each face has a number of *sides* (*külg*).

- Number of sides f_1 — 4, f_2 — 3, f_3 — 8, f_4 — 5.
- I.e. if an edge has the same face in “both sides”, then this edge countes as two sides of that face.
- The number of sides of all faces equals the double of the number of edges.

Theorem (Euler). Let G be a connected planar graph. Define

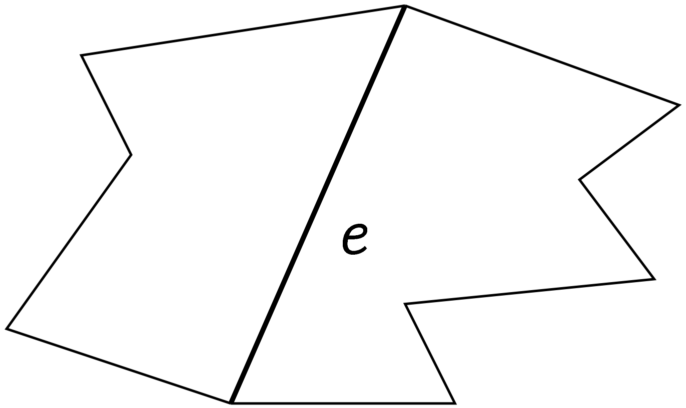
- n — the number of vertices of G,
- m — the number of edges of G,
- f — the number of faces of some drawing of G.

Then $n + f - m = 2$.

Proof. Induction over m.

Base. G is a tree. Then $n = m + 1$ and $f = 1$. Thus $n + f - m = m + 1 + 1 - m = 2$.

Step. Let G be a connected graph that is not a tree. Let G have m edges. There is an edge e whose removal does not disconnect G.



The number of edges and faces in the graph $G-e$ is one less than in G. By induction assumption, $n+(f-1)-(m-1) = 2$. Hence $n + f - m = 2$. □

Corollary. Let G be a planar graph. Define

- n — the number of vertices of G,
- m — the number of edges of G,
- f — the number of faces of some drawing of G.
- k — the number of connected components of G.

Then $n + f - m = k + 1$.

Proof. Apply the previous theorem to each connected component of G. Pay attention to count the infinite face only once. □

Corollary. If G is a *simple* connected planar graph with *at least* 3 *vertices*, then $m \leq 3n - 6$ (same definitions of m and n as before).

Proof. Each face of a drawing of such G has at least 3 sides. Each edge occurs as a side of a face twice, hence

$$2m = \sum_{F \text{ is a face}} \langle \text{number of sides of } F \rangle \geq 3f \quad .$$

Euler's formula gives

$$2 = n + f - m \leq n + \frac{2}{3}m - m = \frac{3n - m}{3}$$

or $3n - m \geq 6$. $\square$

Corollary. K_5 is not planar.

Tõestus. The graph K_5 has $n = 5$ and $m = 10$. If K_5 were planar, then it had to have $m \leq 3n - 6$ or $10 \leq 9$. □

Corollary. If G is a simple connected planar graph with at least 3 vertices, and if G contains no cycles of length 3, then $m \leq 2n - 4$.

Proof. Each face of a drawing of such G has at least 4 sides. Each edge occurs as a side of a face twice, hence $2m \geq 4f$. Euler's formula gives

$$2 = n + f - m \leq n + \frac{1}{2}m - m = \frac{2n - m}{2}$$

or $2n - m \geq 4$. $\square$

Corollary. $K_{3,3}$ is not planar.

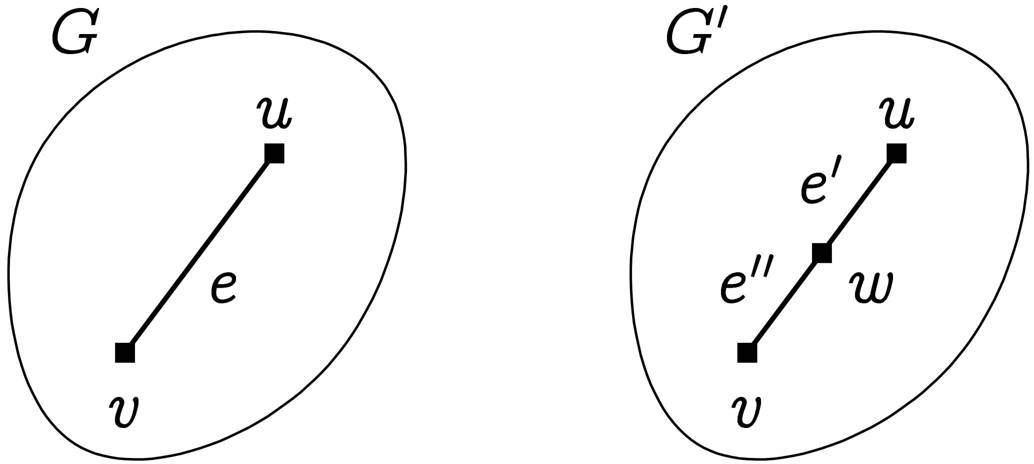
Tõestus. The graph $K_{3,3}$ has $n = 6$ and $m = 9$. It does not contain cycles of length 3. If $K_{3,3}$ were planar, then it had to have $m \leq 2n - 4$ or $9 \leq 8$. □

Corollary. A simple planar graph contains a vertex of degree at most 5.

Proof. Let G be a connected component of a simple planar graph. Assume contrarywise, that the degrees of all vertices of G are at least 6.

Each edge is incident to two vertices, hence $6n \leq 2m$ or $m \geq 3n$. But before we had $m \leq 3n - 6$. $\square$.

Subdividing (*poolitamine*) of an edge: ($G \Longrightarrow G'$):



Edge e is replaced by a vertex w and edges e', e''.

Graphs G_1 and G_2 are *homeomorphic* (*homöomorfsed*), if there is a graph G, such that both G_1 and G_2 can be obtained from G by subdividing edges.

Theorem (Kuratowski). A graph is planar iff it has no subgraphs homeomorphic to K_5 or $K_{3,3}$.

Hence a graph is non-planar iff it "contains" K_5 or $K_{3,3}$ in the following sense:

- The vertices of K_5 or $K_{3,3}$ are the vertices of G.
- The edges of K_5 or $K_{3,3}$ are paths in G.
- Those paths do not intersect each other, except at their common end-vertices.

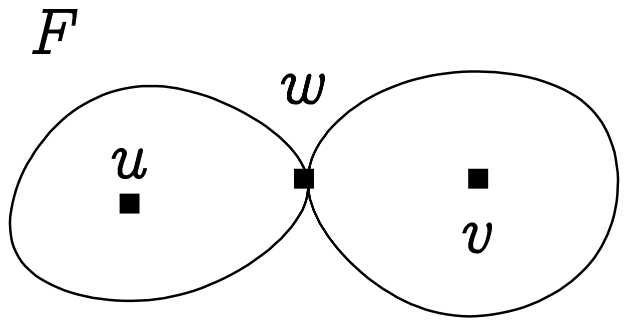
Proof. Assume the contrary — there exist non-planar graphs that "contain" neither K_5 or $K_{3,3}$. Let G be such a graph, with minimal number of edges and no isolated vertices.

G obviously satisfies the following:

- G is a simple graph.
- G is connected.
- G has no bridges.
- G has no cut-vertices.

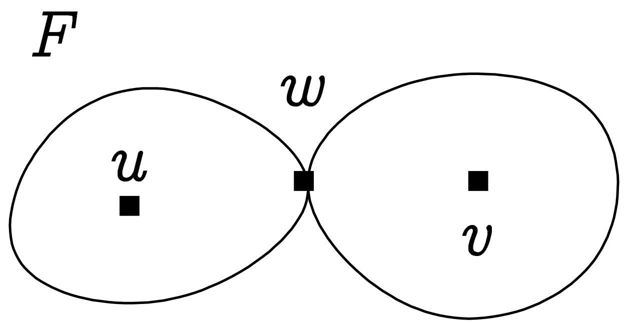
Let e be an edge of G, let $\mathcal{E}(e) = \{u, v\}$. Let $F = G - \{e\}$. Then F is planar, because it satisfies the claims of the theorem and contains no K_5 or $K_{3,3}$.

Claim 1. Graph F contains no vertex w, such that F has the form

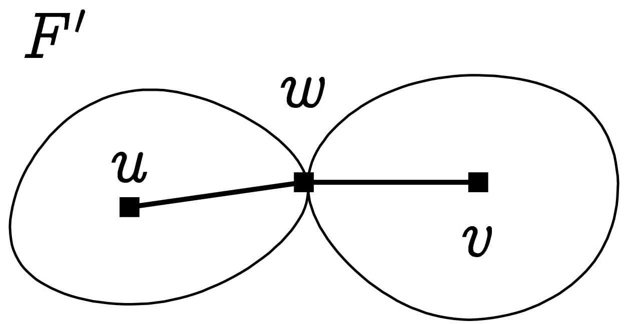


i.e. w is a cut-vertex of F whose removal separates u and v.

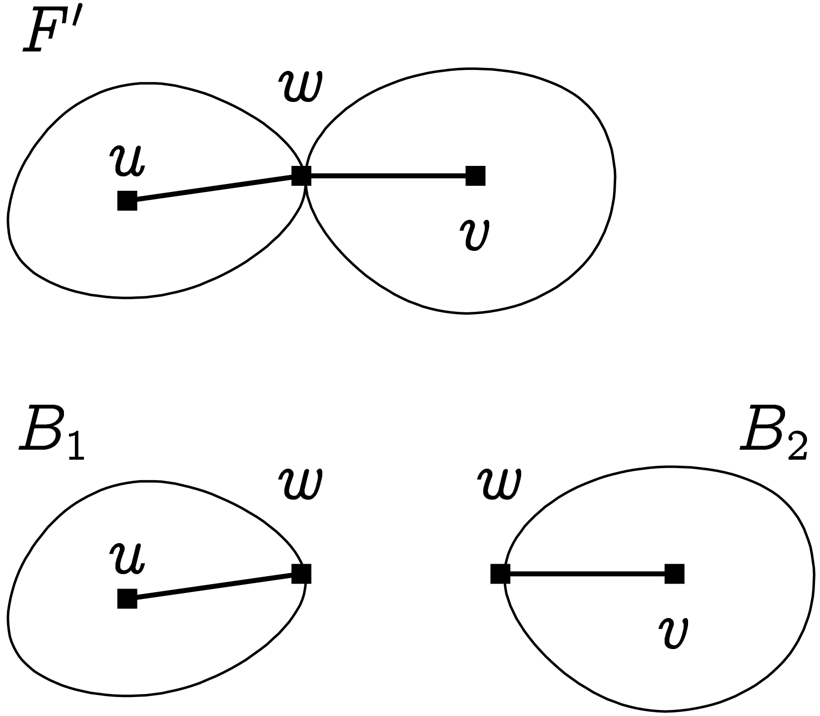
Assume contrarywise that F has the shape



Let F' be obtained from F by adding the following two edges to it:



Let B_1 and B_2 be the following graphs:



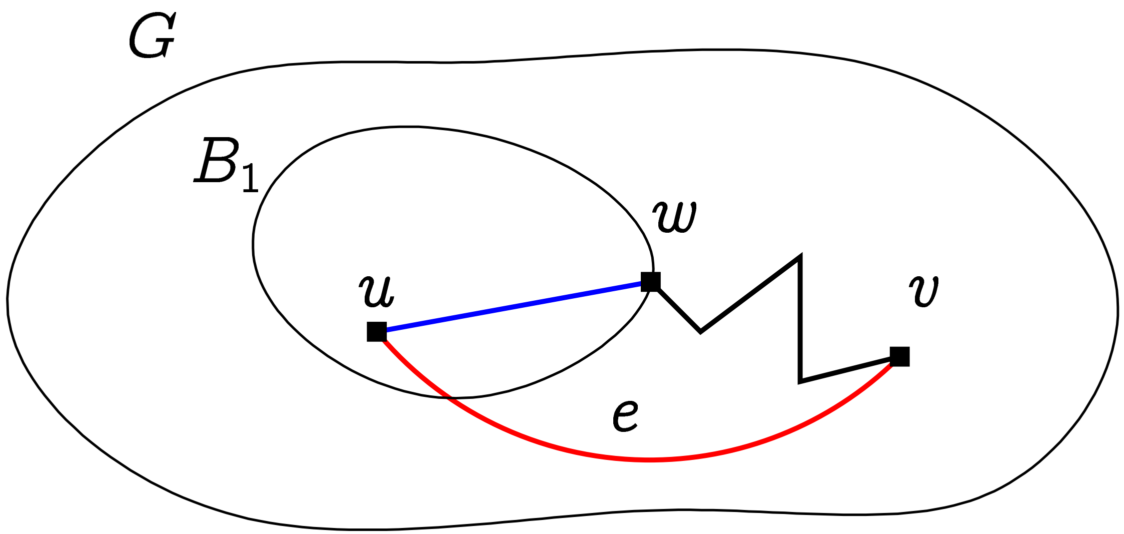
The graphs B_1 and B_2 have less edges than G, hence they satisfy the claim of the theorem.

There are two possibilities:

1. possibility. B_1 (or B_2) contains K_5 or $K_{3,3}$.

This containment must use the new edge between u and w.

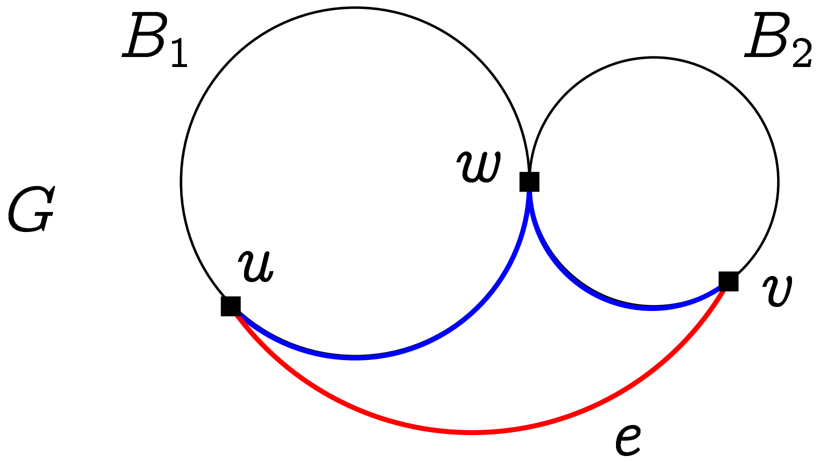
But then also G contains K_5 or $K_{3,3}$:



New edge can be replaced by a path that is outside of B_1.

2. possibility. Both B_1 and B_2 are planar.

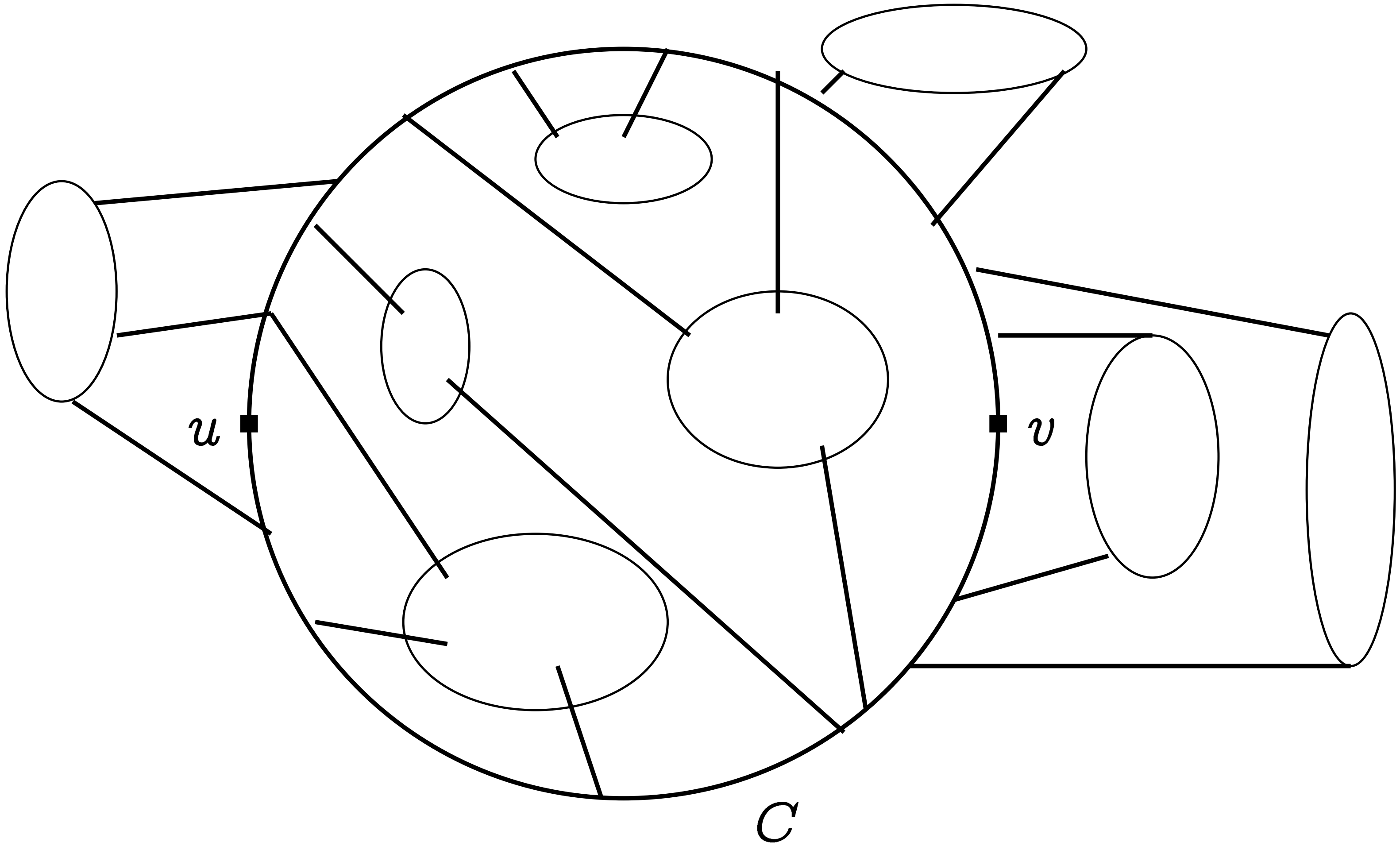
Then G is planar, too. Draw B_1 and B_2 so, that the new edges were on the infinite face:



Claim 1 has been proved.

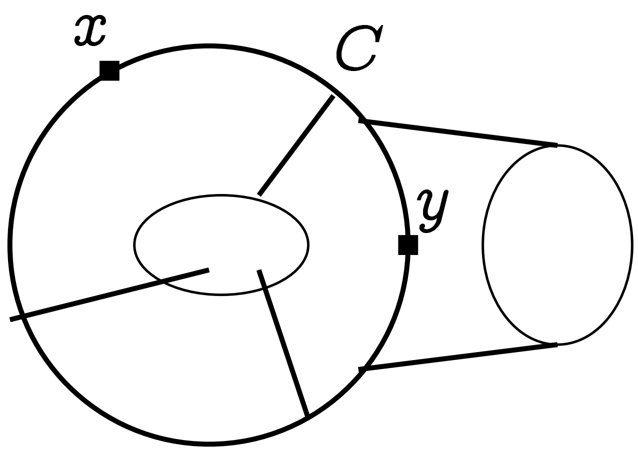
Hence F contains a block containing both u and v. Hence F contains a cycle passing through both u and v.

Draw F on the plane and choose a cycle C passing through u and v in such a way, that the *number of faces* located *inside* C is *as large as possible*.

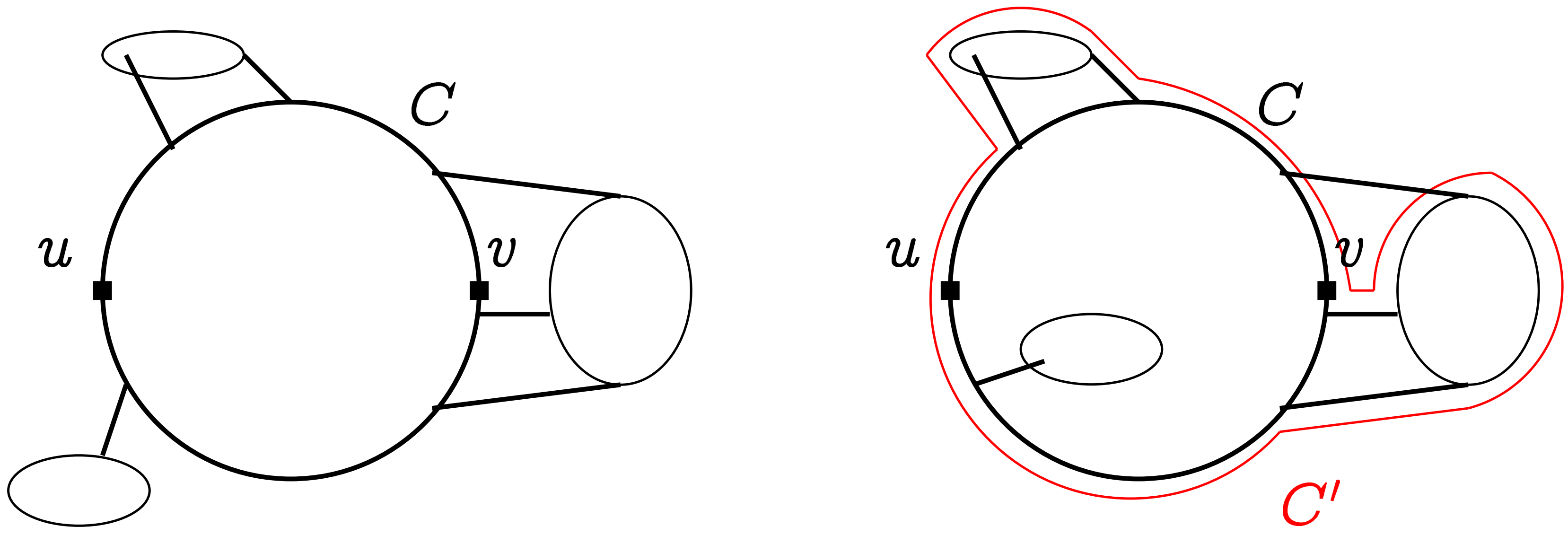


Besides the cycle C, the graph F contains more *components*. Some of them are *inner*, the others are *outer*.

Let x and y be vertices on C. Some inner/outer component *separates* x and y if it is on the way of drawing a line from x to y inside/outside C.

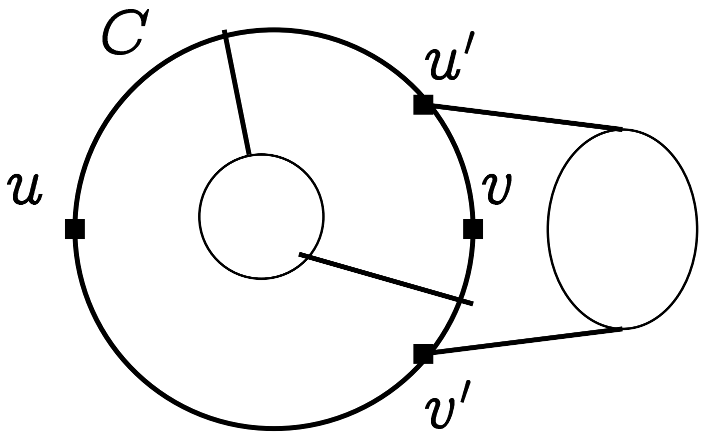


Claim: all outer components separate u and v and are connected to C with exactly two edges:

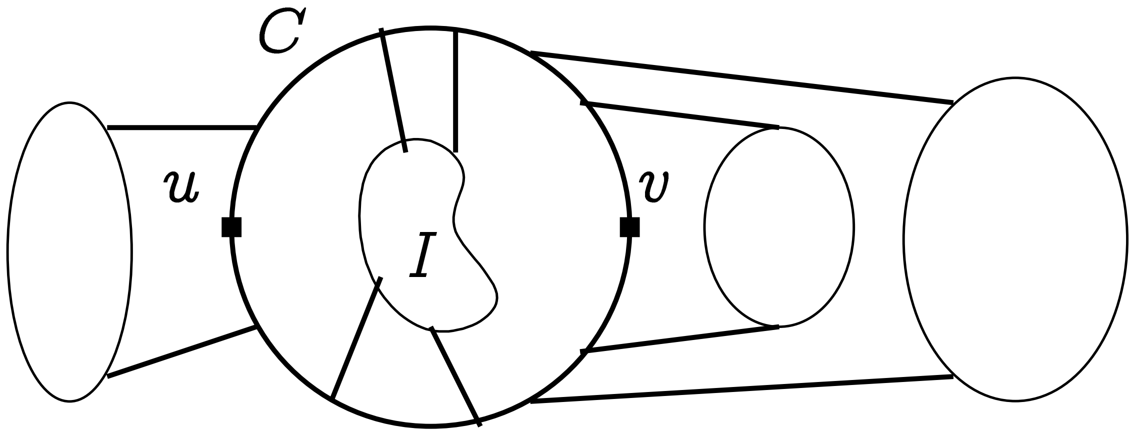


Otherwise there is a drawing / cycle that puts more faces inside C.

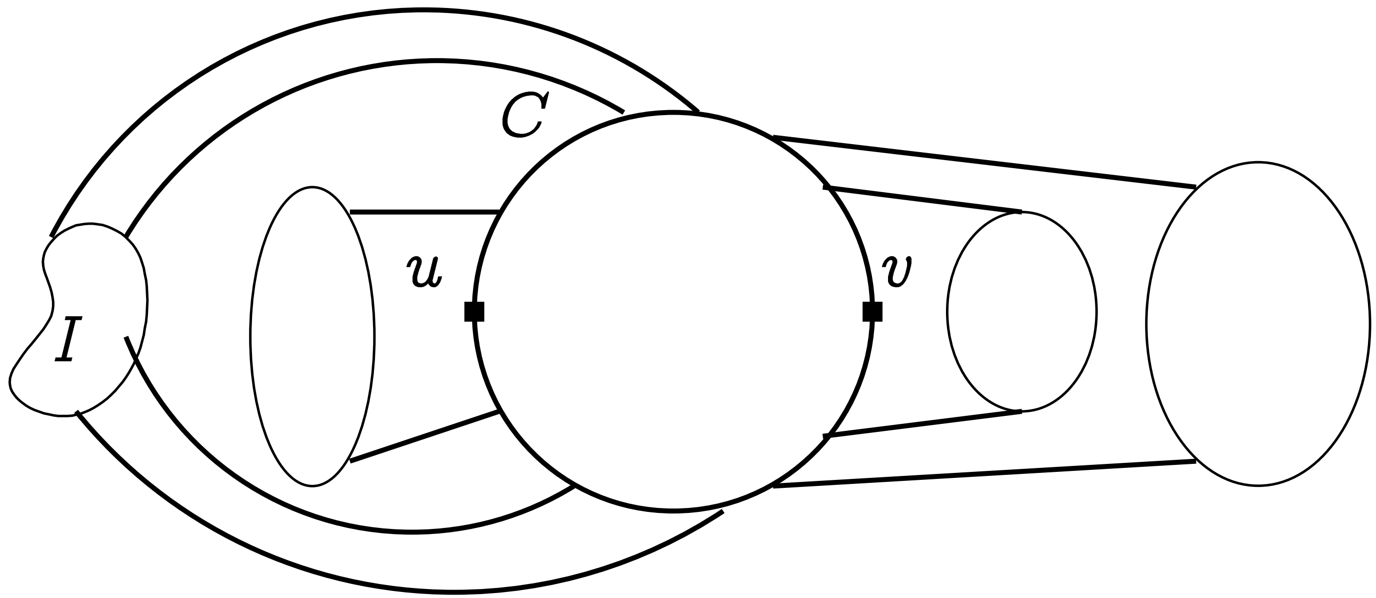
Claim 2. There exist an inner component and an outer component (attached to C at vertices u' and v'), such that this inner component separates both u and v, and u' and v'.



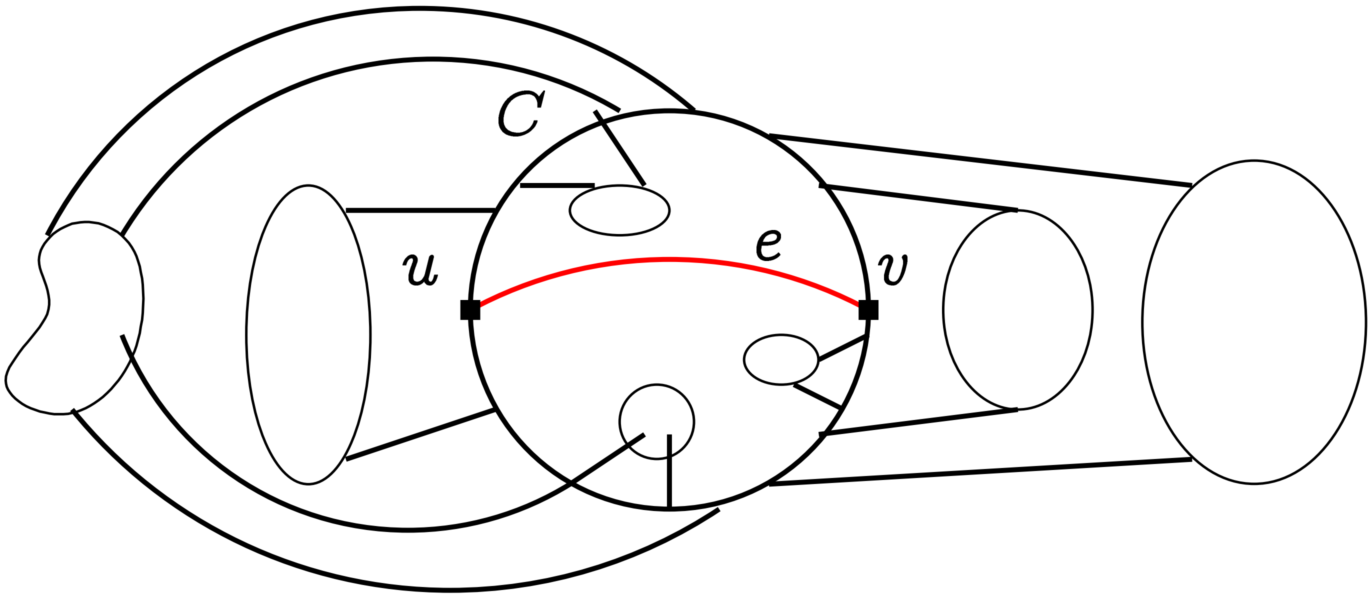
Proof of the claim: let I be an inner component separating u and v, that for no outer components separates the vertices where this outer component attaches to C:



We can move I outside:

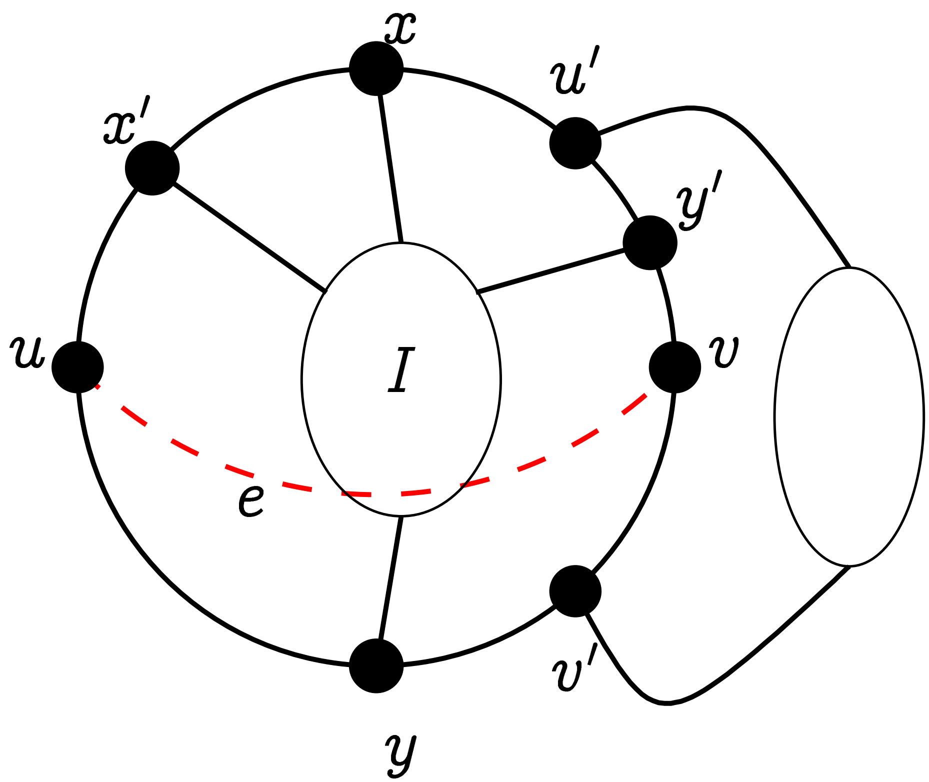


If claim 2 was wrong, then we can move out all inner components that separate u and v. Afterwards we can re-add the edge e to the graph F, giving us the graph G. This gives us a planar drawing of G. Hence the claim 2 must hold.



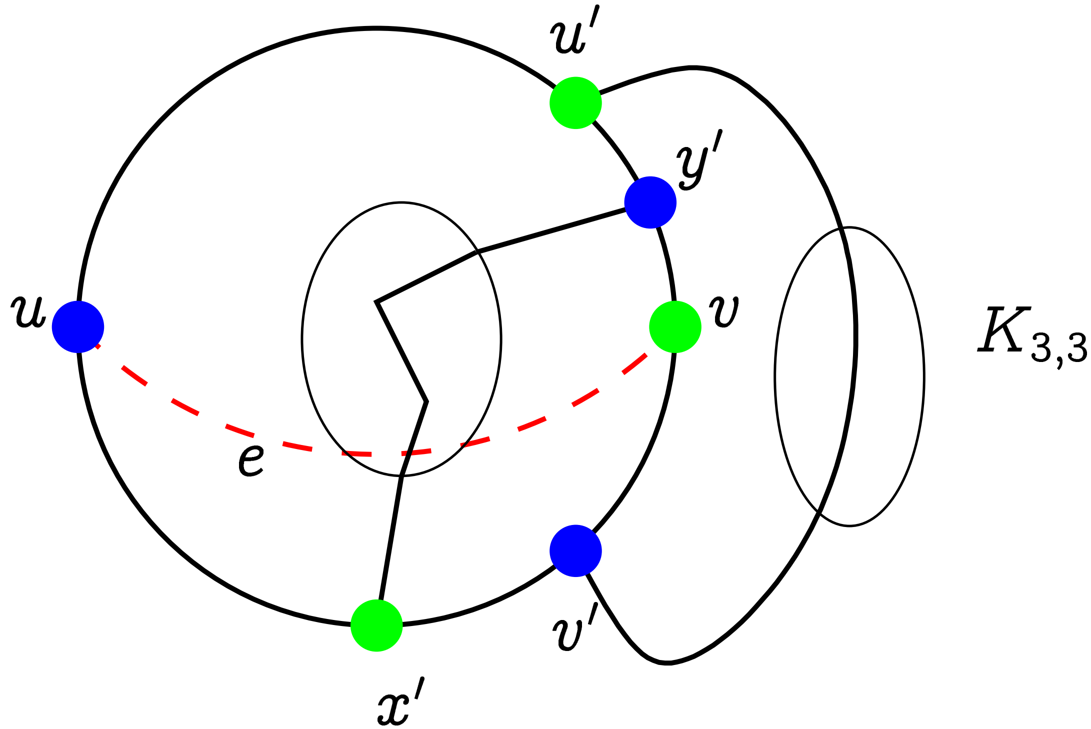
Let x, y be the vertices that I has separating u and v.

Let x', y' be the vertices that I has separating u' and v'.



They can be arranged in several ways. We will consider them and find K_5 or $K_{3,3}$ from G in all cases.

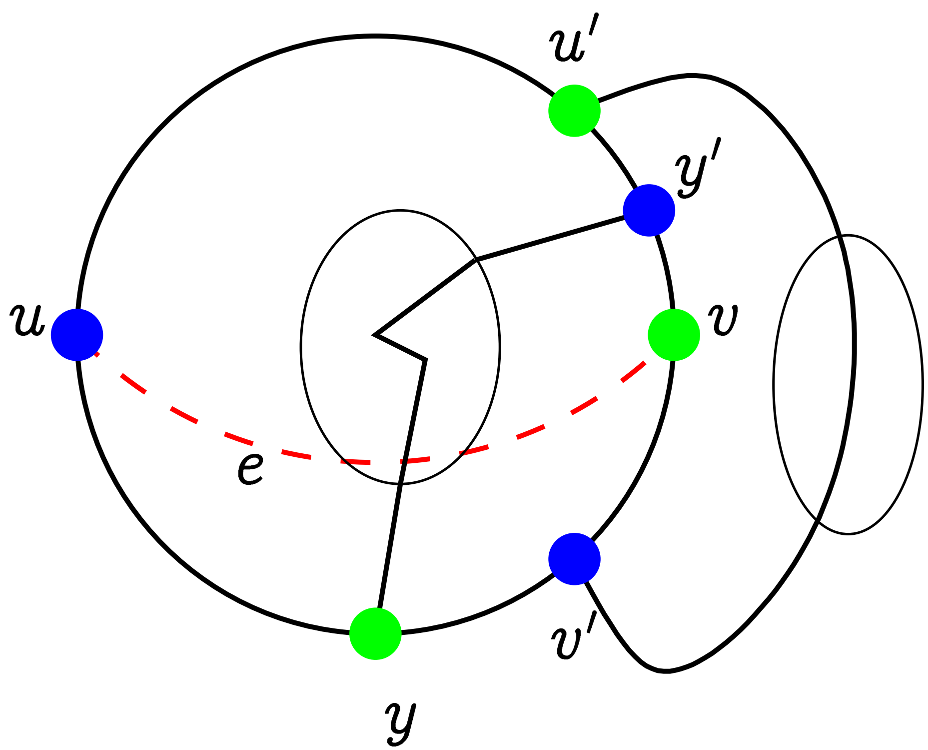
1st way. x', y' differ from u and v and I separates u and v due to x', y' as well.



2nd way x', y' differ from u and v and I does not separate u and v due to x', y'.

We can assume that x', y' are on the same side as x.

1st option. y is between u and v'.

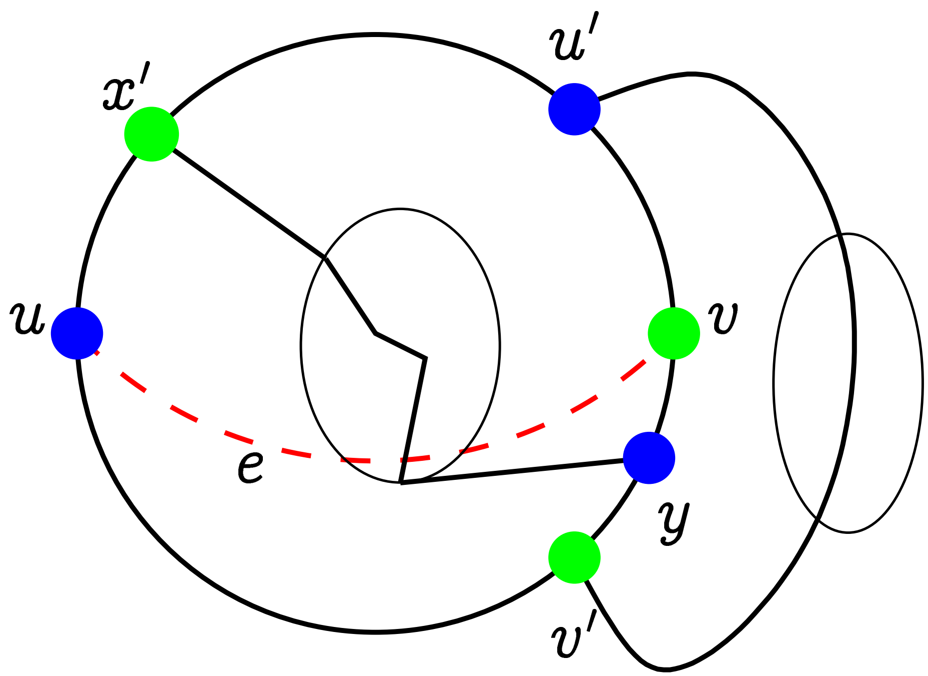


$K_{3,3}$

2nd way. x', y' differ from u and v and I does not separate u and v due to x', y'.

We can assume that x', y' are on the same side as x.

2nd option. y is between v' and v.

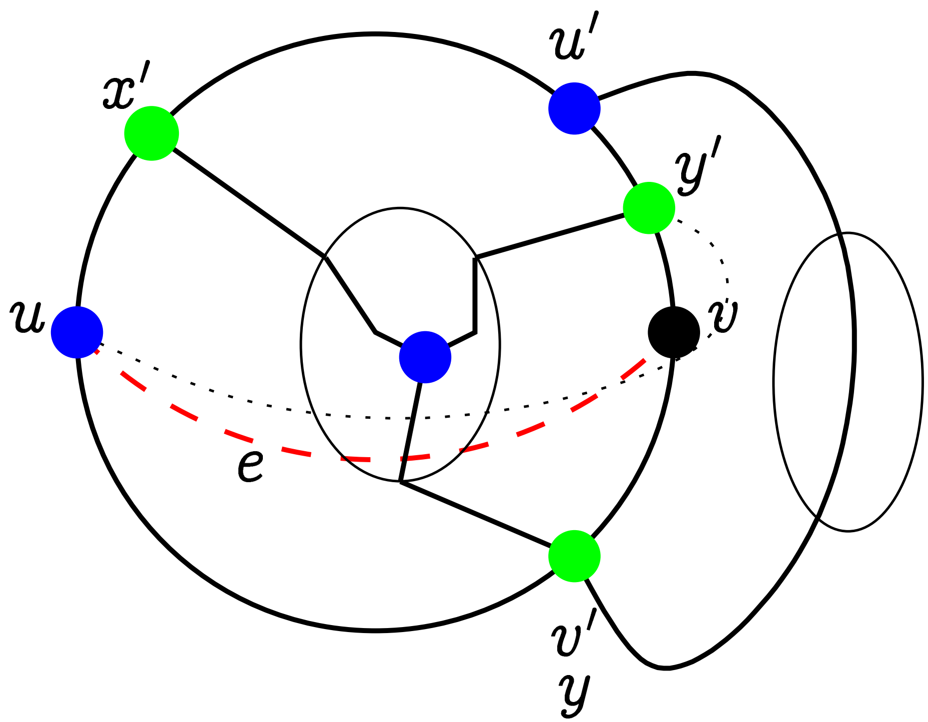


$K_{3,3}$

2nd way. x', y' differ from u and v and I does not separate u and v due to x', y'.

We can assume that x', y' are on the same side as x.

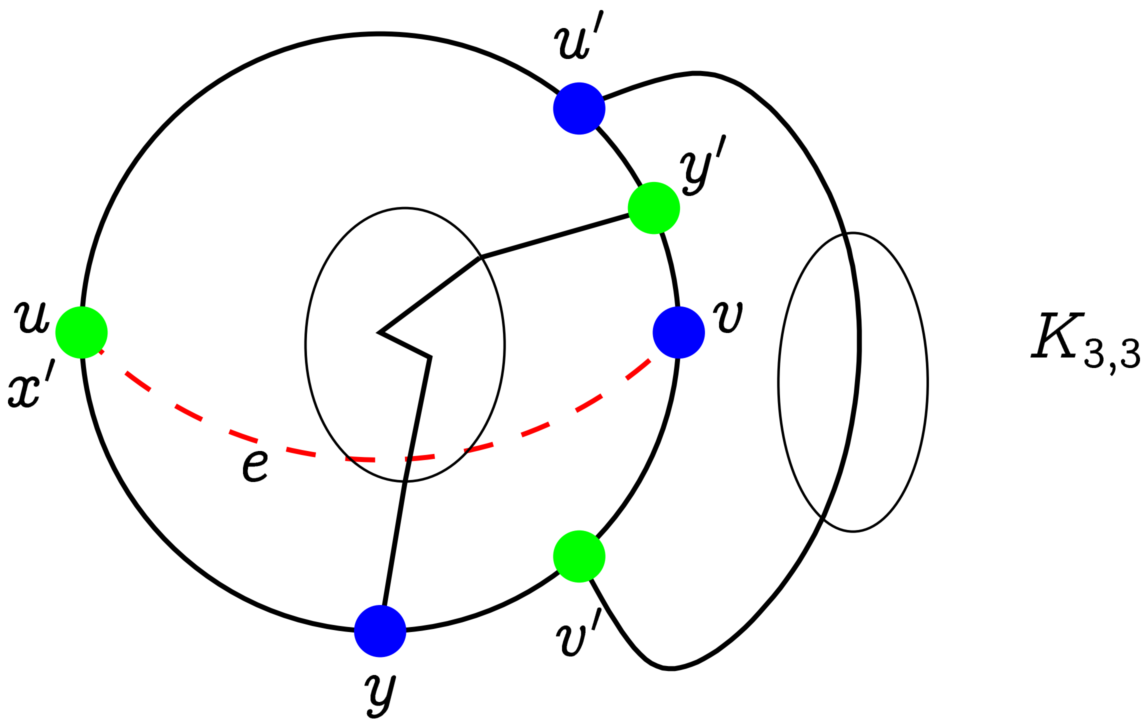
3rd option. $y = v'$.



$K_{3,3}$

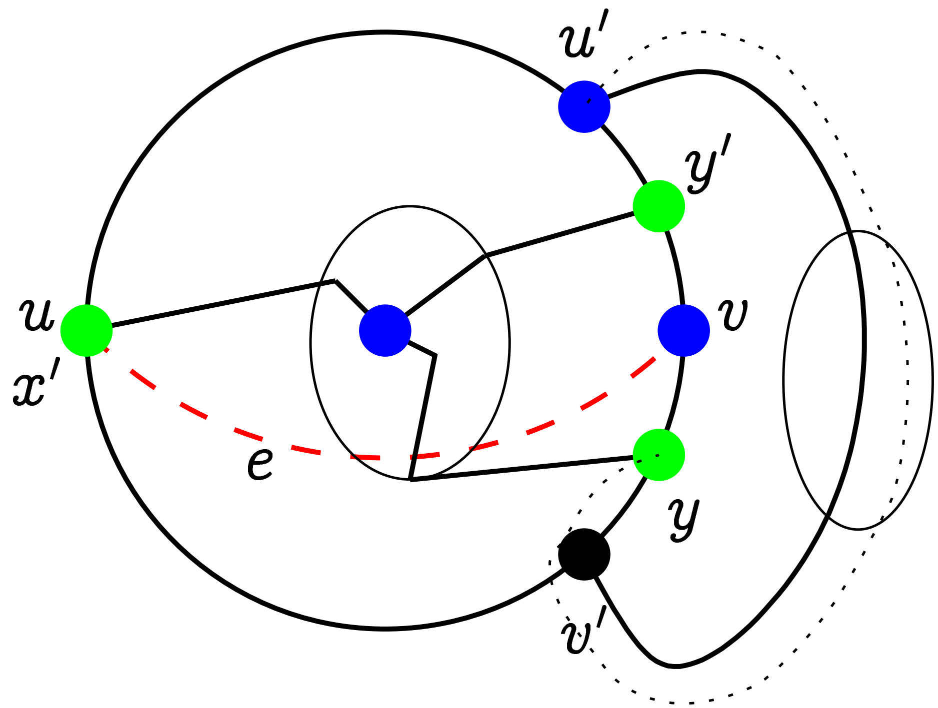
3rd way. $x' = u$ and $y' \neq v$. Assume that y' is between u' and v.

1st option. y is between u and v'.



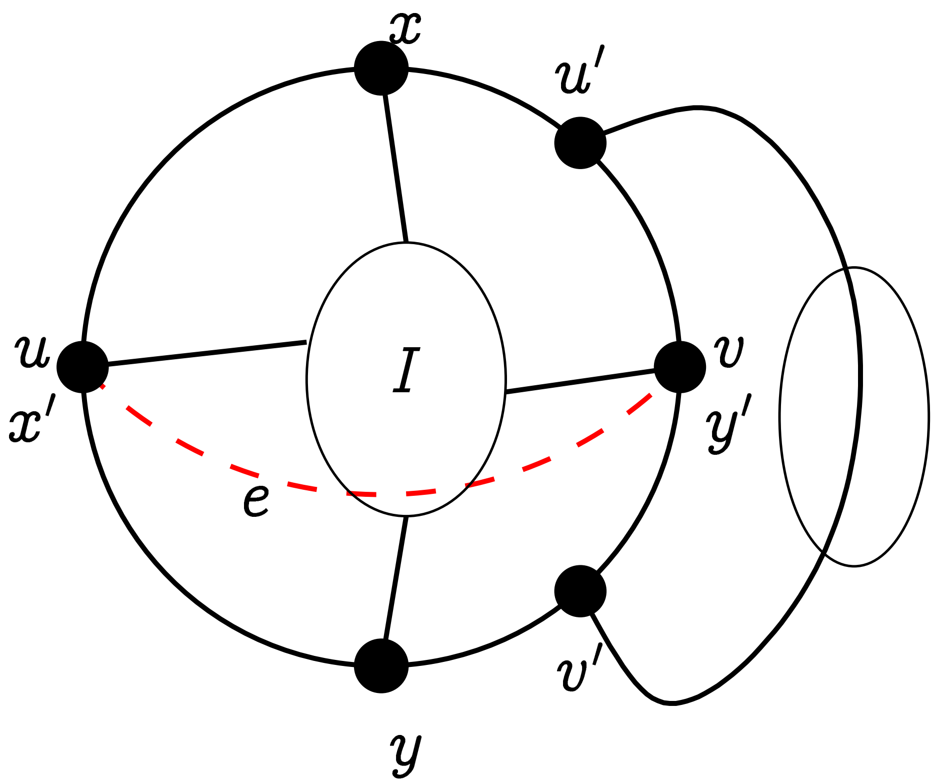
3rd way. $x' = u$ and $y' \neq v$. Assume that y' is between u' and v.

2nd option. y is between v' and v or $y = v'$.



$K_{3,3}$

4th way. $x' = u$ and $y' = v$.

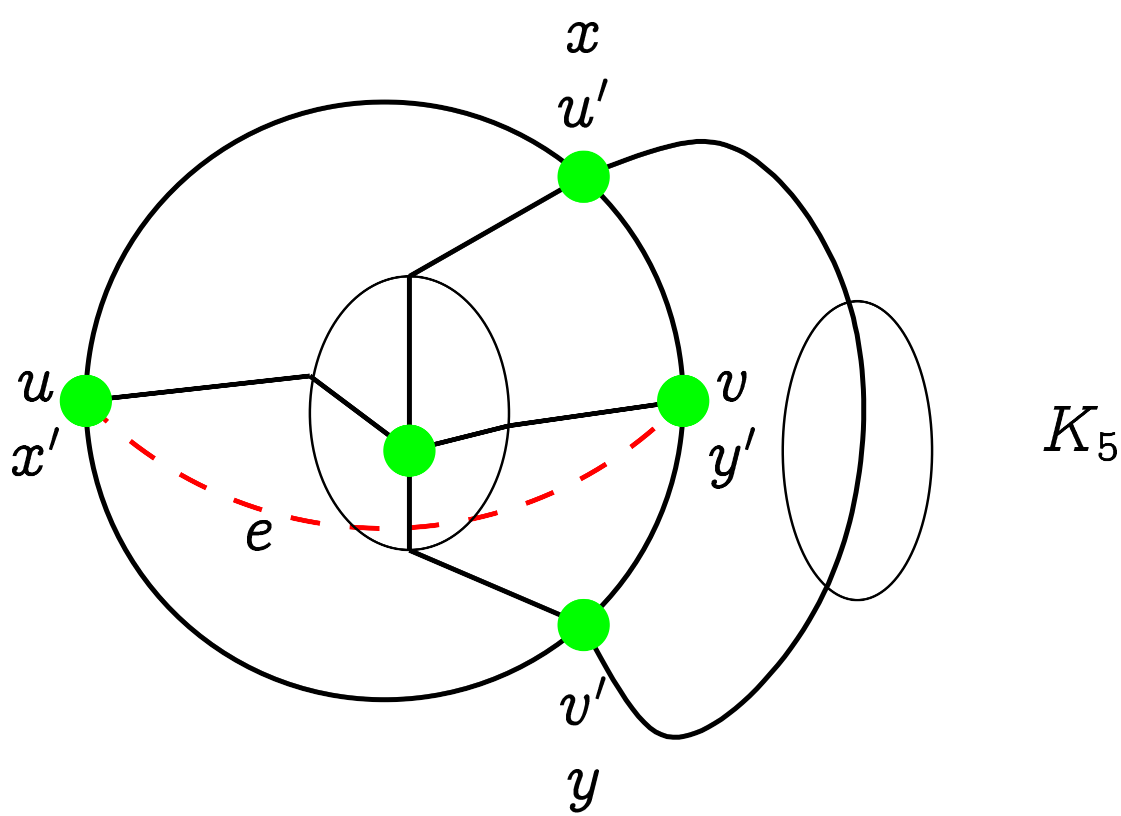


If x and y are not u' and v', then we exchange the notations ($u \leftrightarrow u'$, $v \leftrightarrow v'$, $x \leftrightarrow x'$, $y \leftrightarrow y'$, $e \leftrightarrow$ the path outside C). We are back to one of the three first ways.

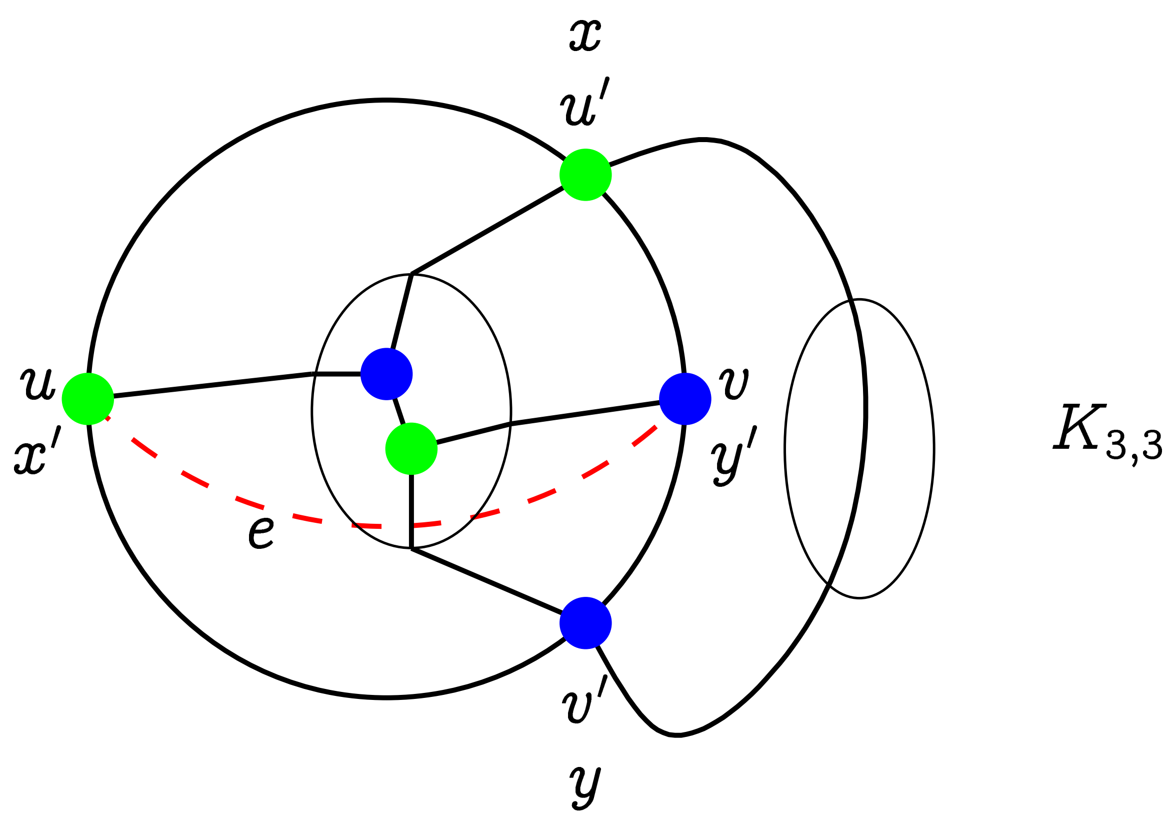
We are left with the case $x' = u$, $y' = v$, $x = u'$, $y = v'$.

The vertices neighbouring u, v, u', v' within the inner component are connected somehow within the component.

The first possible connection:

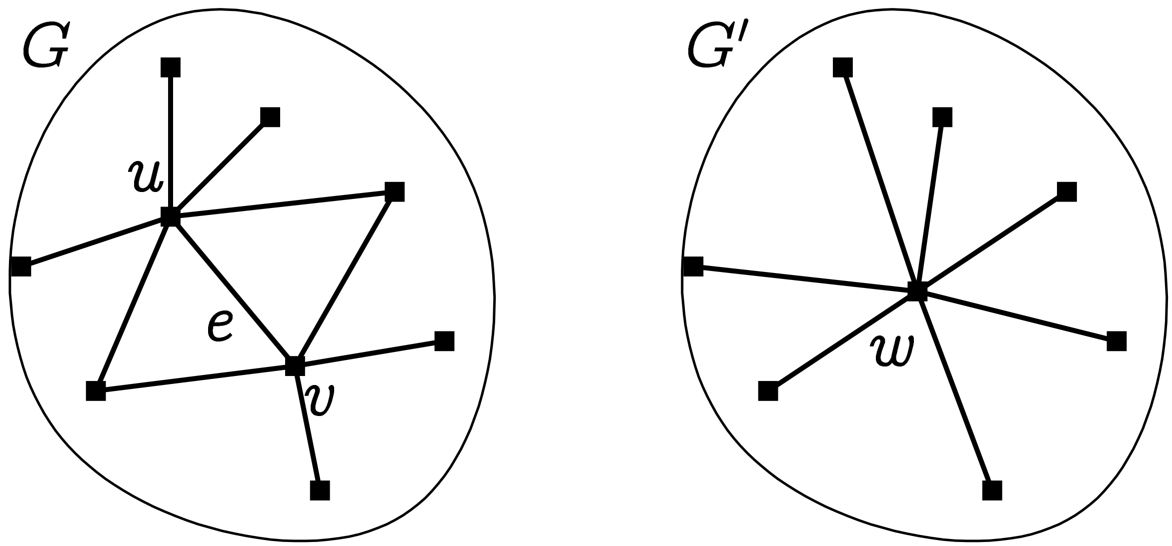


The second possible connection:



The theorem is proven. □

Edge *contraction (kokkutõmbamine)* ($G \Longrightarrow G'$):



When edges are contracted, a planar graph remains planar.

Theorem (Wagner). A graph is planar iff it has no subrgaphs contractible to K_5 or $K_{3,3}$.

Proof. If G is planar, then all its subrgaphs are planar. If we contract edges in a planar subgraph, we still get a planar graph, thus we can't get K_5 or $K_{3,3}$.

If G is not planar then there exists $H \leq G$ such that H is homeomorphic to K_5 or $K_{3,3}$. Contracting the edges we can reverse the effect of subdividivision. $\square$