# Matchings and coverings

Consider a set $X$. We want to pair its elements.

The set of potential pairs is constrained by the relation

$$P \subseteq \{\{x, y\} \mid x, y \in X, x \neq y\},$$

showing which elements can be paired.

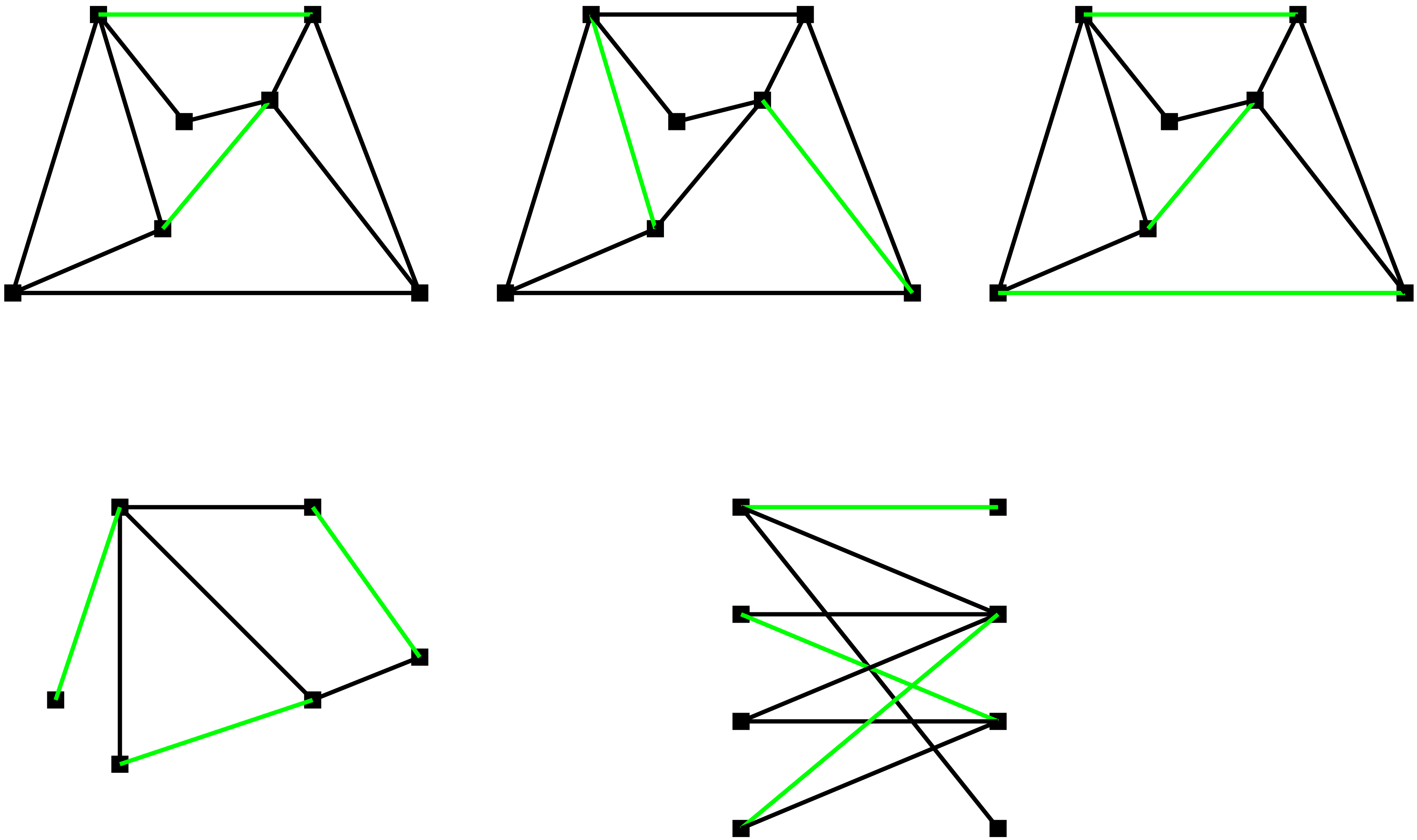
In this lecture, we assume $(X, P)$ to be a simple graph.

Often $(X, P)$ is a bipartite graph. E.g., $X$ can be the set of lecture halls and potential times of particular lectures. $P$ can indicate which halls can accommodate which lectures.

Let $G = (V, E)$ be a simple graph. *Matching (kooskõla)* in the graph $G$ is a set $M \subseteq E$ of edges such that for each $v \in V$ we have $\deg_M(v) \leq 1$.

The matching is *maximal* if its cardinality is the largest possible.

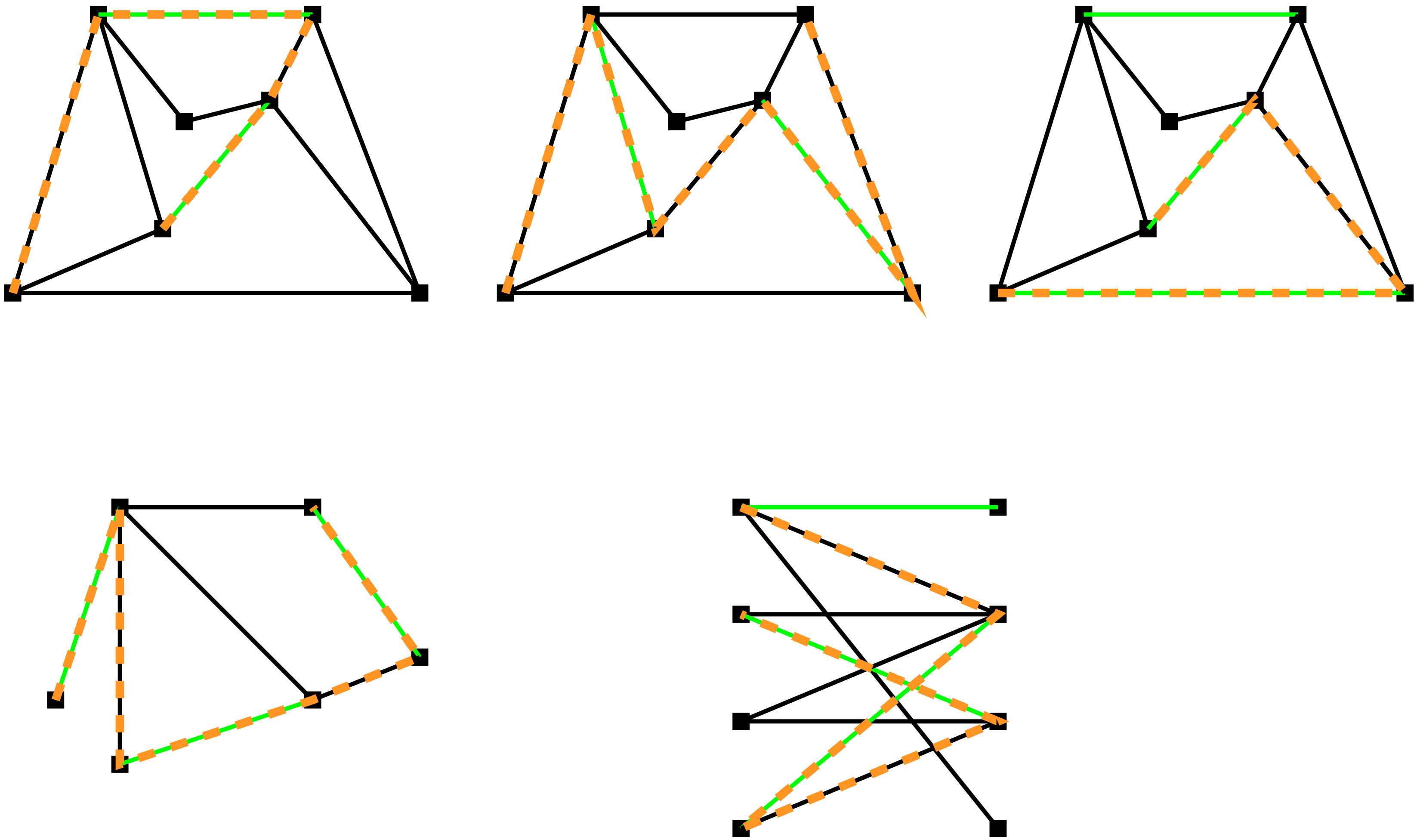
The matching $M$ is *perfect (täielik)* if $\deg_M(v) = 1$ holds for every $v \in V$.



Let $G = (V, E)$ be a simple graph, $M \subseteq E$ a matching and $P$ some path (with different endpoints) in the graph $G$.

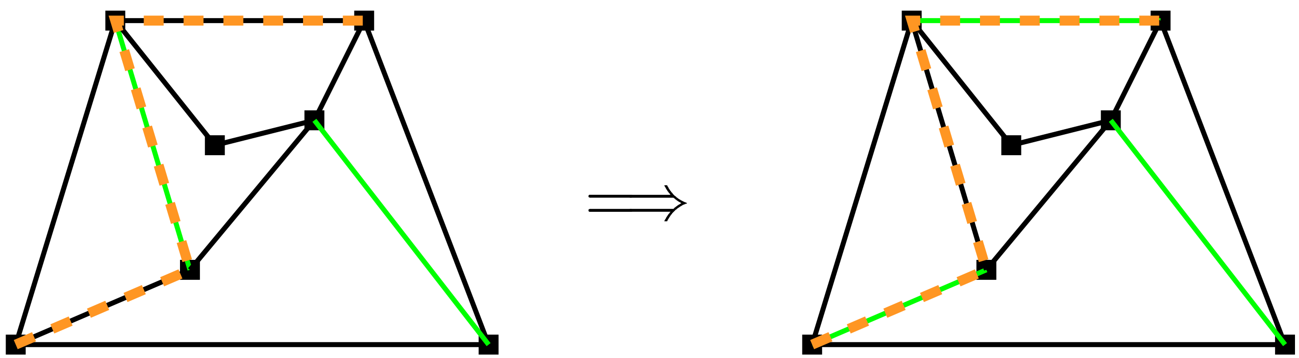
The path $P$ is *$M$-alternating (vahelduv)* if its edges alternately belong to the sets $M$ and $E \setminus M$.

The path $P$ with endpoints $x$ and $y$ is *$M$-extensible (laienev)* if it is $M$-alternating and $\deg_M(x) = \deg_M(y) = 0$.



**Theorem (Berge).** Matching $M$ in the graph $G = (V, E)$ is maximal iff there are no $M$-extensible paths in $G$.

Proof $\Rightarrow$. Assume to the contrary that there exists an $M$-extensible path $P$ in $G$.



Consider $P$ as a set of edges.

Let $M' = (M \backslash P) \cup (P \backslash M)$. Then $|M'| = |M| + 1$.

It is easy to verify that $M'$ is a matching. Let $v \in V$, we will show that $\deg_{M'}(v) \leq 1$. There are three options.

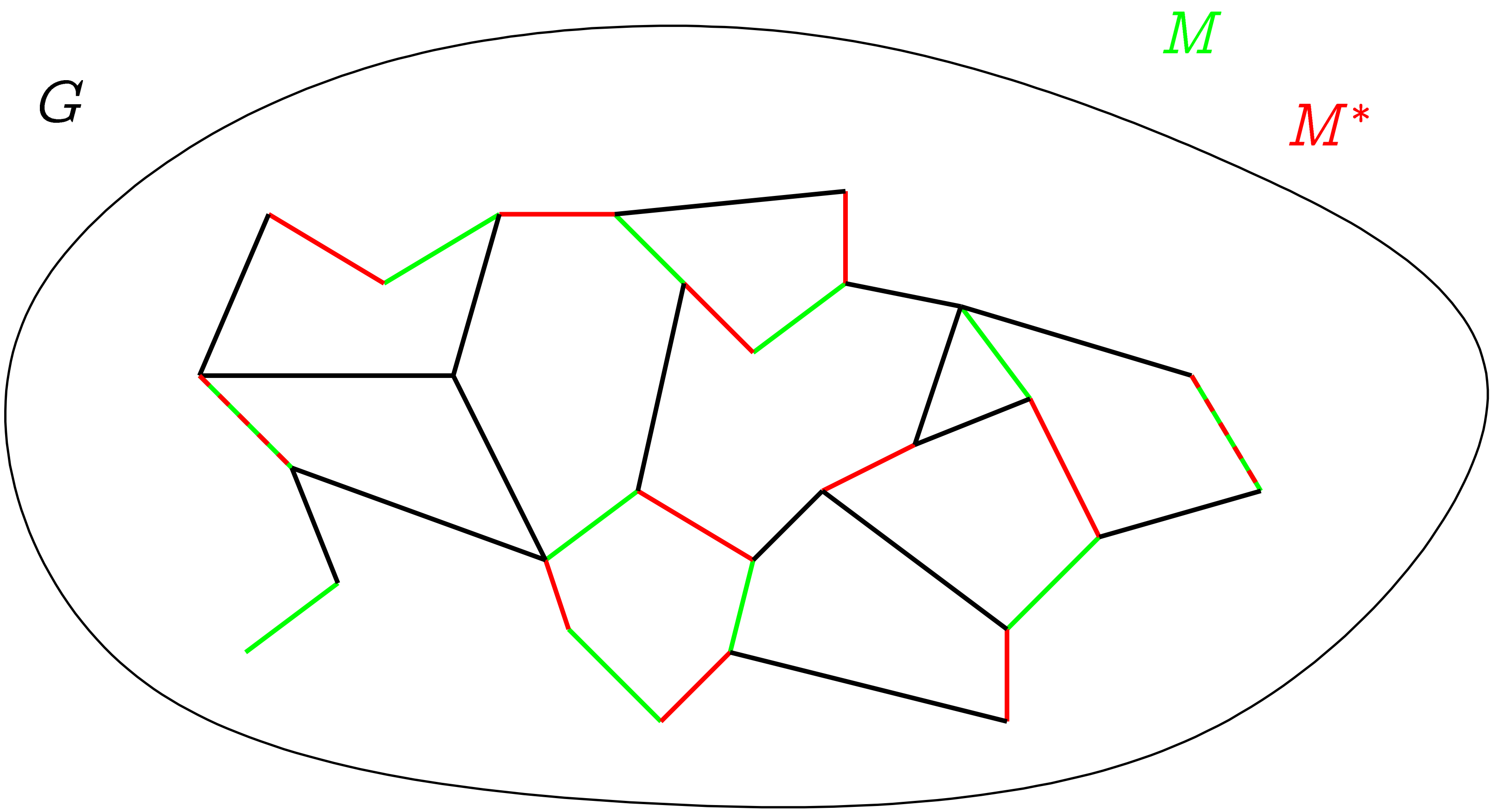
- $v$ is not on the path $P$. Then $\deg_M(v) = \deg_{M'}(v)$. Indeed, let $e \in E$ be incident with $v$. As $e \notin P$, we have $e \in M \Leftrightarrow e \in M'$.

- $v$ is an endvertex of $P$. Then $\deg_{M'}(v) = \deg_M(v)+1 = 1$.

- $v$ is an internal vertex of $P$. Then $\deg_{M'}(v) = \deg_M(v) = 1$.

Proof $\Leftarrow$. We will construct an $M$-extensible path.

Let $M^*$ be a maximal matching in $G$. Then $|M| < |M^*|$.
Consider the graph $H = (V, M \cup M^*)$.



For each $v \in V$ we have $\deg_H(v) \leq 2$. Possible connected components of $H$ are:

- Isolated vertices.
- Paths.
  - Closed paths, i.e. cycles.
    - The edges of $M$ and $M^*$ alternately.
  - Open paths. Options:
    - A lonely edge $e \in M \cap M^*$.
    - The edges of $M$ and $M^*$ alternately. Options:
      - Having one end in $M$, another end in $M^*$.
      - Having both ends in $M$.
      - Having both ends in $M^*$.

Since $|M| < |M^*|$, there must exist a connected component of $H$ having more edges from $M^*$ than edges from $M$.

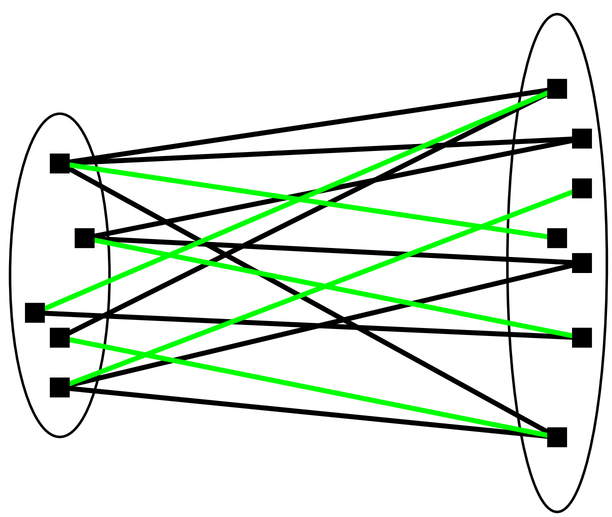
The only such components are open paths having both ends in $M^*$.

These paths are $M$-extensible. $\square$

Let $G = (V, E)$ be a graph and let $S \subseteq V$. *Neighbourhood (naabrus)* of $S$ is the set

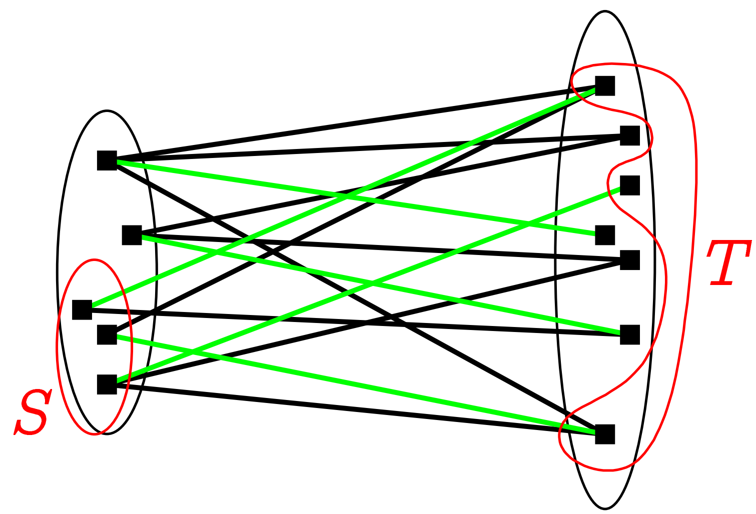
$$N(S) = \{w \mid w \in V, \exists e \in E, \exists v \in S : \mathcal{E}(e) = \{v, w\}\} \ .$$

**Theorem (Hall).** Let $G = (V, E)$ be a bipartite graph with vertex set partition to $X$ and $Y$. The graph $G$ has a matching $M$ with the property $\forall x \in X : \deg_M(x) = 1$ iff for each $S \subseteq X$ the inequality $|N(S)| \geq |S|$ holds.



Proof $\Rightarrow$. Let $M$ be a matching with the required property. Let $S \subseteq X$. Consider the set

$$T = \{y \mid y \in Y, \exists x \in S : (x, y) \in M\} \ .$$

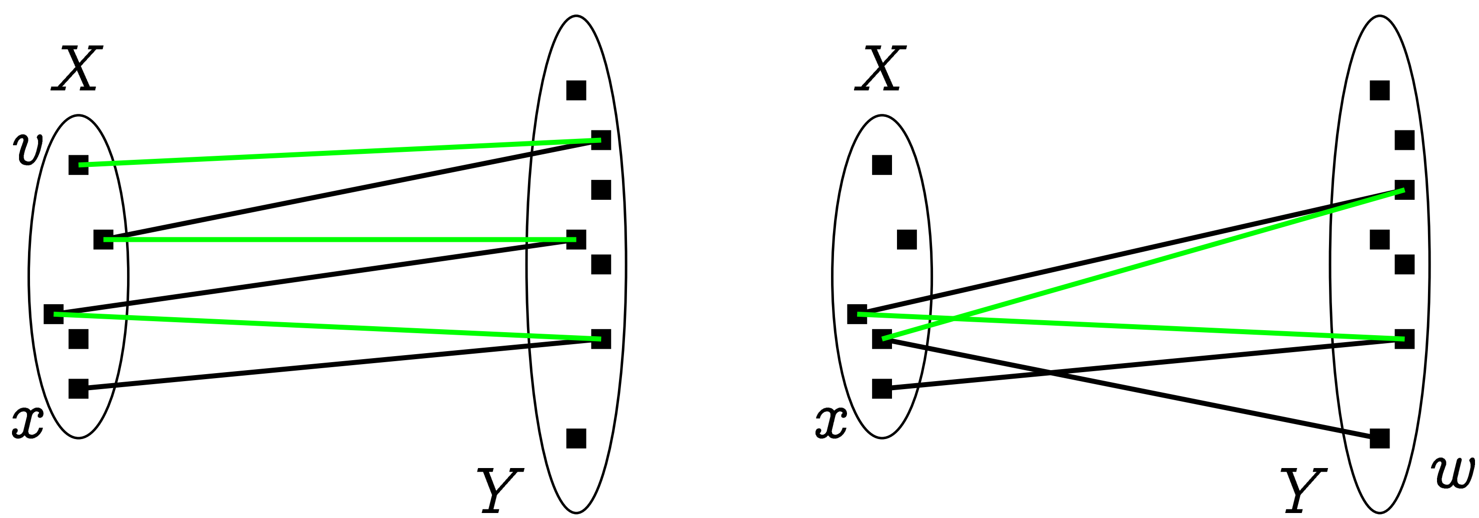



Then $|T| = |S|$, since each $x \in S$ defines a different $y$. We also have $T \subseteq N(S)$, consequently $|S| = |T| \leq |N(S)|$.

Proof $\Leftarrow$. Let $M$ be some maximal matching. Assume to the contrary that there exists $x \in X$, such that $\deg_M(x) = 0$.

Let $S \subseteq X$ be the set of all vertices $v \in X$ such that there exists an $M$-alternating path from $x$ to $v$. Note that $x \in S$.

Let $T \subseteq Y$ be the set of all vertices $w \in Y$ such that there exists an $M$-alternating path from $x$ to $w$.



We will show that

I. $N(S) = T$;

II. $|S \backslash \{x\}| = |T|$.

As a consequence, we will get a contradiction:

$$|N(S)| = |T| = |S \backslash \{x\}| = |S| - 1 < |S| \ .$$

*Part I.* Let $v \in S$ and let $P$ be an $M$-alternating path from $x$ to $v$. Note that the last edge on the path $P$ belongs to $M$.

Let $w \in Y$ be a neighbour of vertex $v$. There are two options:

1. $w$ is on the path $P$. The part of $P$ from $x$ to $w$ is an $M$-alternating path from $x$ to $w$. Thus $w \in T$.

2. $w$ is not on the path $P$. Two options again:

   - $(v, w) \in M$. Then $(v, w)$ is the last edge on the path $P$, because there are no other edges in $M$ incident with $v$. Thus we are back to the 1st option.

   - $(v, w) \notin M$. Then $P$ together with the edge $(v, w)$ is an $M$-alternating path from $x$ to $w$. Thus $w \in T$.

*Part II*. We will construct a bijection between $S \backslash \{x\}$ and $T$.

Let $v \in S \backslash \{x\}$. Then there is an edge $e \in M$ incident with $v$ (the last edge on the $M$-alternating path from $x$ to $v$). We let the other endvertex $w$ of $e$ to correspond to $v$. We proved on the last slide that $w \in T$.

Let $w \in T$. If there was no edge $e \in M$ being incident with $w$, we would get an $M$-extensible path from $x$ to $w$. Berge Theorem forbids this, thus we have such an edge $e$.

We let the other endvertex $v$ of $e$ to correspond to $w$. Obviously, $v \in S$. Also, $v \neq x$, since the other endvertex of $e$ is not $x$, because $\deg_M(x) = 0$. $\square$

**Corollary.** Regular (i.e. with all vertex gedrees equal) bipartite non-null graph has a perfect matching.

Proof. Let $G = (V, E)$ be a bipartite graph with partition $X$ and $Y$. Let $k > 0$ be the degree of all the vertices. Since

$$|X| \cdot k = \sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y) = |Y| \cdot k,$$

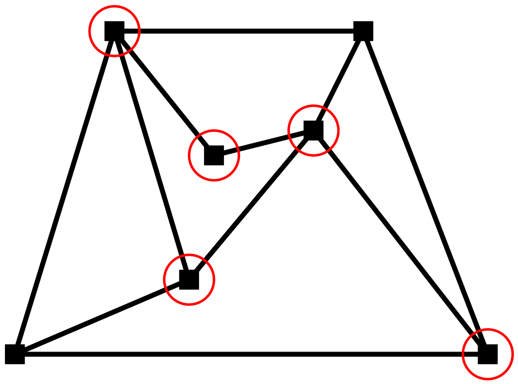
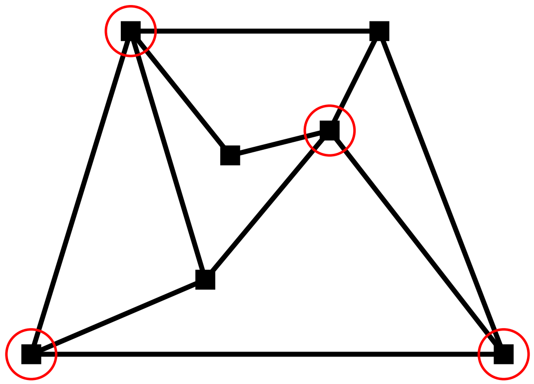
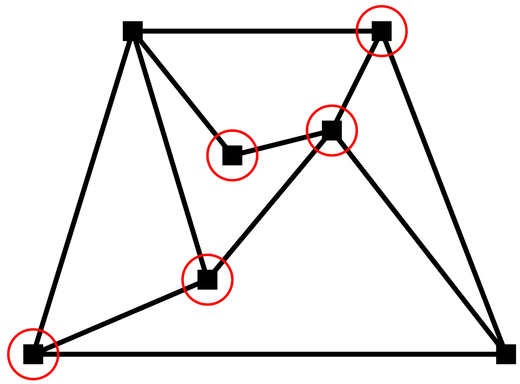
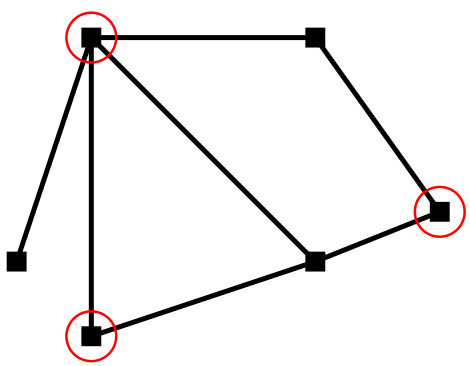
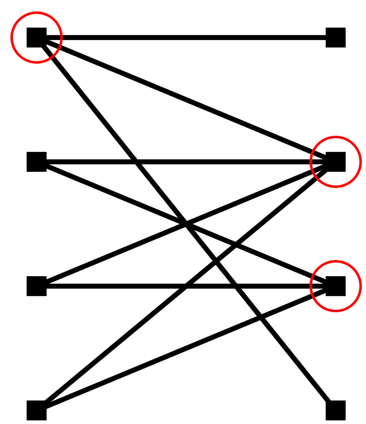
we have $|X| = |Y|$. Let $S \subseteq X$. Since

$$|S| \cdot k = \sum_{x \in S} \deg(x) \leq \sum_{y \in N(S)} \deg(y) = |N(S)| \cdot k,$$

we get $|S| \leq |N(S)|$. Thus there exists a matching $M$ such that $\deg_M(x) = 1$ for each $x \in X$. Since $|X| = |Y|$, we also have $\deg_M(y) = 1$ for each $y \in Y$. $\square$

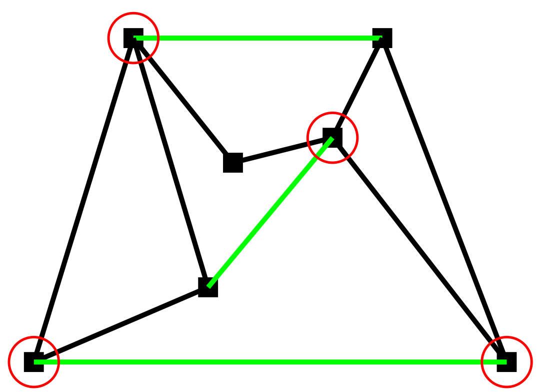
Let $G = (V, E)$ be a simple graph. *Cover (kate)* in graph $G$ is the set $K \subseteq V$ of vertices such that each $e \in E$ is incident with some vertex from $K$.

Cover is *minimal* if its cardinality is the smallest possible.

**Proposition.** Let $G = (V, E)$ be a simple graph, $M$ some of its matchings and and $K$ some of its covers. Then $|M| \leq |K|$.

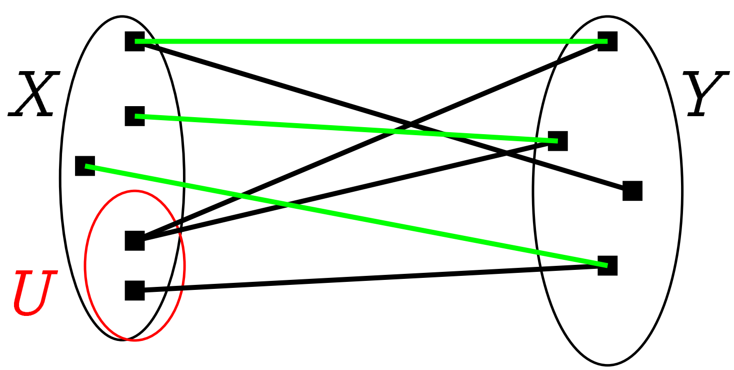
Proof. For each edge $e \in M$, there exists a vertex $v \in K$ such that $e$ is incident with $v$. For different edges these vertices differ, since the edges of $M$ can not have common endvertices. $\square$



**Theorem (König).** Let $G = (V, E)$ be a bipartite graph. Then the cardinalities of maximal matchings and minimal covers are equal.

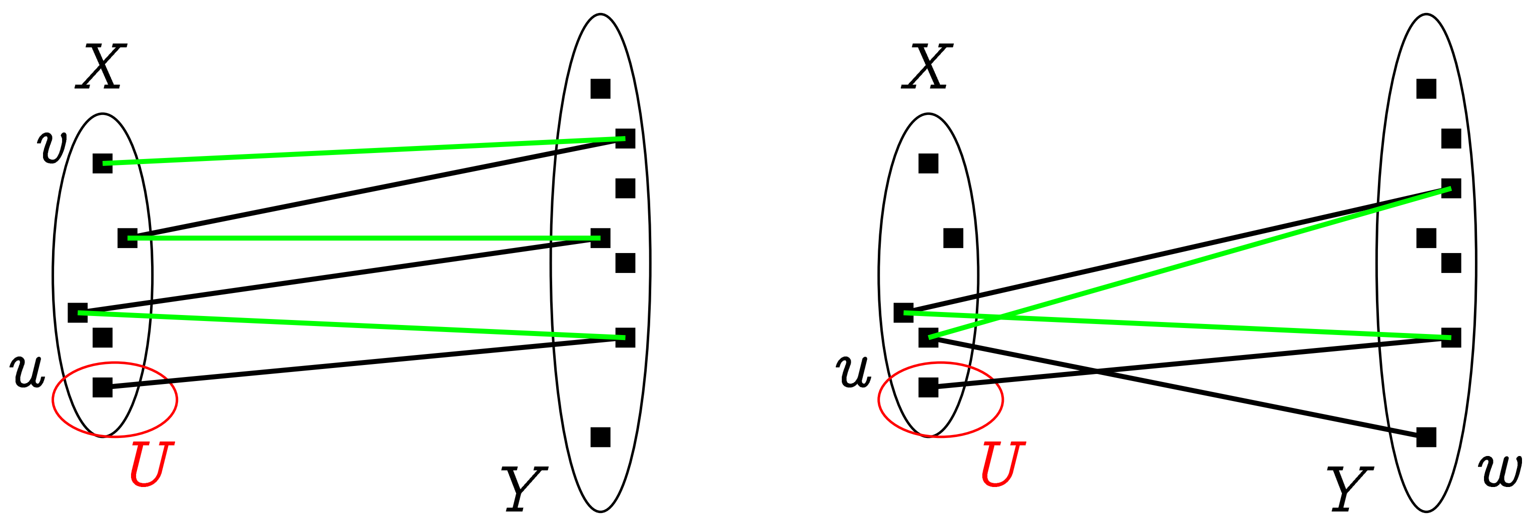
Proof. Let $X$ and $Y$ be the partition of $G$ and let $M$ be one of its maximal matchings. We will construct a cover $K$ such that $|M| = |K|$.

Let $U \subseteq X$ be the set of such vertices $u \in X$ that $\deg_M(u) = 0$. Then $|M| = |X \backslash U|$.



Let $S \subseteq X$ be the set of such vertices $v \in X$ that for some $u \in U$ there exists an $M$-alternating path from $u$ to $v$. Then $U \subseteq S$.

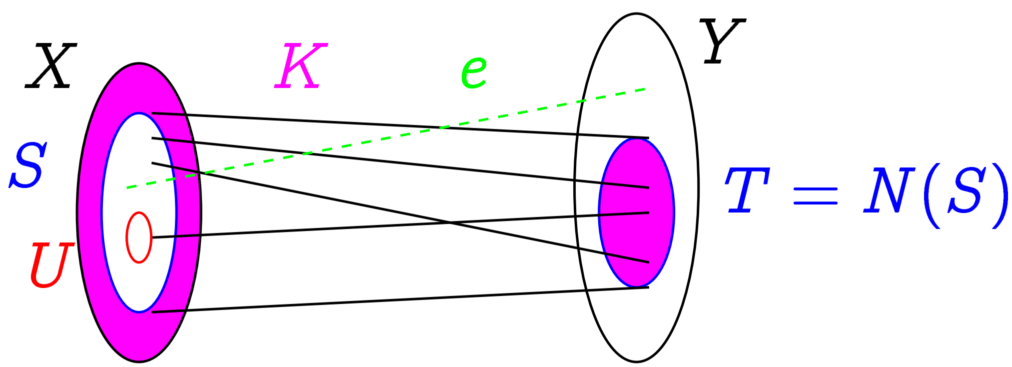
Let $T \subseteq Y$ be the set of all such vertices $w \in Y$ that for some $u \in U$ there exists an $M$-alternating path from $u$ to $w$.




Similarly to the proof of Hall's Theorem we can prove $N(S) = T$ and $|T| = |S \backslash U|$.

Let $K = T \cup (X \backslash S)$. Then $K$ is a cover.

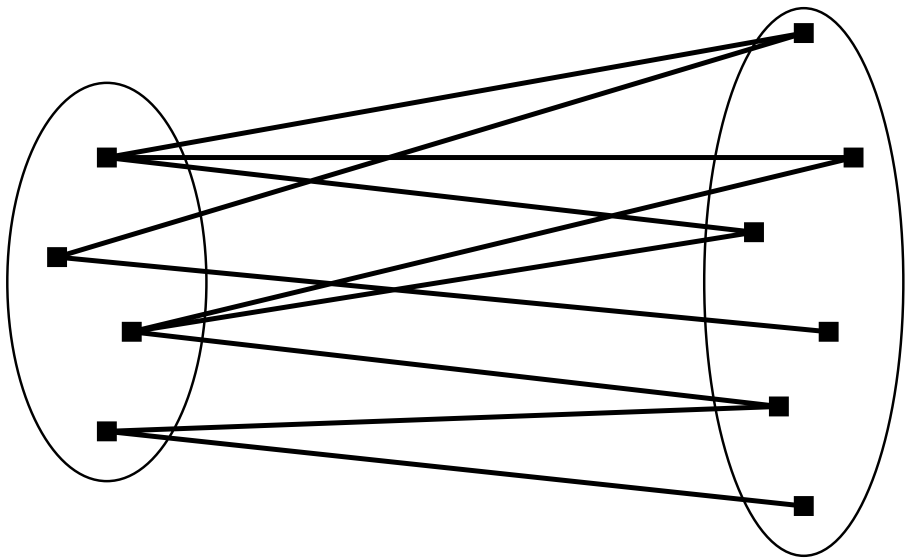
Indeed, assume that there is an edge $e \in E$ that is not incident with any vertex of $K$. Then one endvertex of $e$ is in $S$ and another one in $Y \backslash T$. Contradiction with the observation $N(S) = T$.



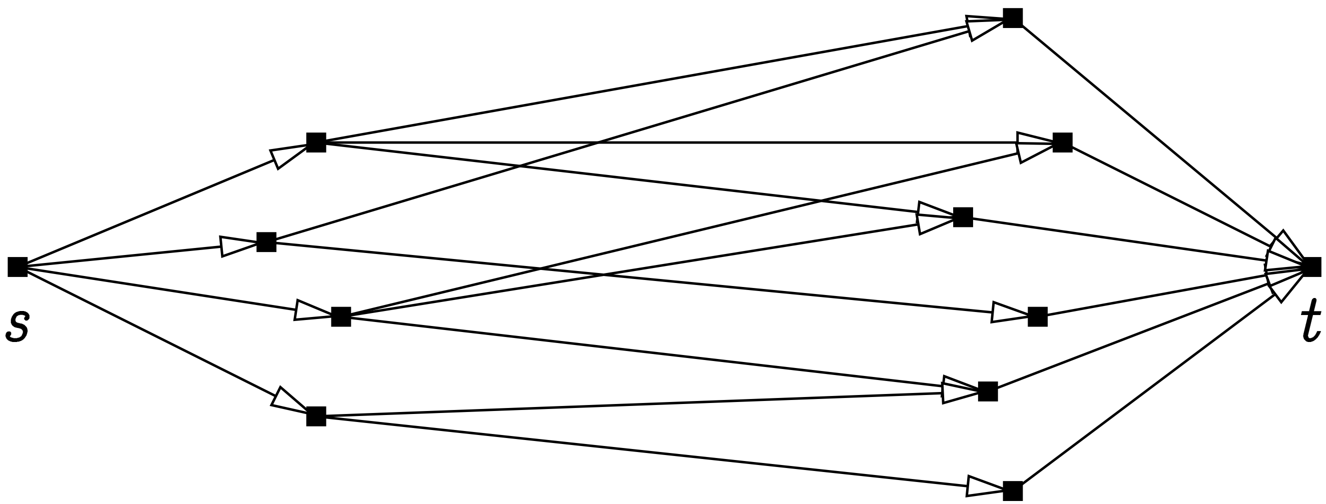

$$|K| = |T| + |X \backslash S| = |S \backslash U| + |X \backslash S| = |X \backslash U| = |M| \ .$$

$\square$

How to find maximal matchings in bipratite graphs?



find the maximal flow




the capacities of all arcs are 1

Ford-Fulkerson algorithm allows us to find the maximal flow, assigning an integer flow to each arc.

A maximal matching in the original graph is given by the edges that were assigned the flow 1.

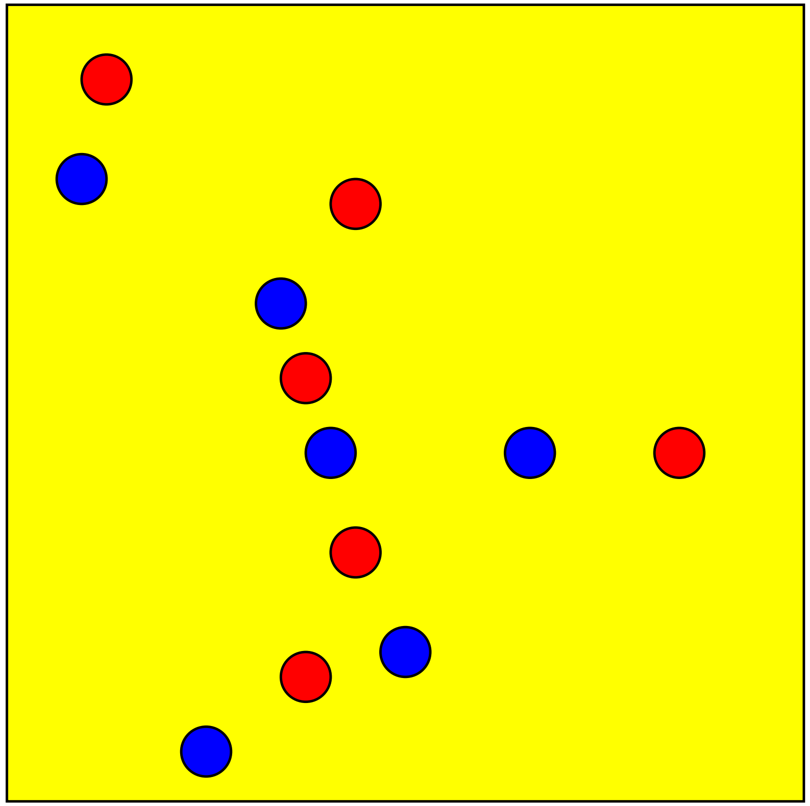
There are problems where

- The edges of a bipartite graph have been assigned costs.
- One has to find the maximal matching having the least cost.

This exercise reduces to finding the minimum-cost maximal flow.

Example: let two pictures of the same slowly-moving objects be given, taken in two different time moments.

Which two blobs correspond to the same object?



Finding the minimal cover in a bipartite graph:

- First find a maximal matching.
- Then look at our proof of König's theorem. It is constructive.