Double counting

- Consider an agricultural experiment. Suppose it is desired to compare the yield of v different varieties of grain.

- It is quite possible that there would be an interaction between the environment (type of soil, rainfall, drainage, etc.) and the variety of grain which would alter the yields.

- So, b blocks (sets of experimental plots) are chosen in which the environment is fairly consistent throughout the block.

- We are interested in comparing all the pairs of grains in a fair manner.

- Growing every variety in a plot in every block, may, for large experiments be too costly or impractical.
- One could use smaller blocks which do not contain all of the varieties.
- To assure fair comparison and to minimize the effects of chance due to incomplete blocks, we would want to design the blocks so that the probability of two varieties being compared (i.e. are in the same block) is the same for all pairs. This property would be called *balance* in the design.
- Statistical techniques, in particular Analysis of Variance (ANOVA), could then be used to reach conclusions about the experiment.

A *balanced incomplete block design* (BIBD) is a set X of $v \geq 2$ elements called *varieties* and a collection of $b > 0$ subsets of X, called *blocks*, such that the following conditions are satisfied:

- each block consists of exactly k varieties, $v > k > 0$,
- each variety appears in exactly r blocks, $r > 0$,
- each pair of varieties appear simultaneously in exactly λ blocks, $\lambda > 0$.

This structure is also known as (v, b, r, k, λ)-design.

The parameters of the design can not be arbitrary.

Prove that the folowing relations hold in a (v, b, r, k, λ)-design:

- $bk = vr$
- $r(k-1) = \lambda(v-1)$

Proof of $bk = vr$. Consider the set of pairs

$$\{(x, B) : \text{ variety } x \text{ is contained in block } B\}$$

and count the number of its elements.

On one hand, there are v possible choices for variety x and this x appears in r pairs. Thus the number of pairs is vr.

On the other hand, there are b choices for block B, each one of them containing k varieties. Thus the number of pairs is bk.

Proof of $r(k-1) = \lambda(v-1)$. Fix a particular variety p, consider the set of pairs

$$\{(y, B) : \text{ varieties } p \text{ and } y \text{ are contained in block } B\}$$

and count the number of its elements.

On one hand, there are $v-1$ choices for the variety y and the pair (p, y) appears togehter in λ blocks, so the number of such pairs is $\lambda(v-1)$.

On the other hand, p appears in r blocks and can be paired with $k-1$ other elements in such a block, thus the number of pairs is $r(k-1)$.

$(7, 7, 3, 3, 1)$-design, a.k.a. Fano plane: