Test one

1. Consider the graphs R_n ($n \geq 3$) with the vertex set $V(R_n) = \{0,1\}^n$ and edge set

$$E(R_n) = \{\{\overline{a}, \overline{b}\} \,:\, \text{vectors } \overline{a} \text{ and } \overline{b} \text{ differ in exactly two positions}\}\,.$$

   Which ones of these graphs are Eulerian? Semi-Eulerian? Hamiltonian? Semi-Hamiltonian?
   Solution. Note that vectors with odd Hamming weigth (the number of ones in the vector) are connected only to the vectors with odd weigth and similarly for the vectors of even weigth. Thus these graphs are not connected.

2. Prove that if $n \geq 3$ then $\overline{Q_n}$ is Hamiltonian.
   Solution. We can make use of Ore theorem. Take two non-neighbouring vertices u and v in $\overline{Q_n}$. Their degrees are $2^n - n - 1$, thus

$$\deg(u) + \deg(v) = 2^{n+1} - 2n - 2$$

   and we need to prove the inequality

$$2^{n+1} - 2n - 2 \geq 2^n\,,$$

   which is equivalent to

$$2^n \geq 2n + 2\,.$$

   This holds true for $n = 3$ (as $8 \geq 8$) and since exponential function increases faster than linear, this inequality also holds for larger values of n.

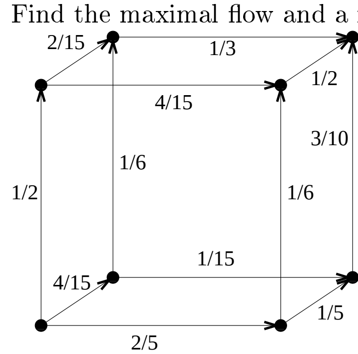
3. Find $|E(L(K_{m,n}))|$.
   Solution. The graph $L(G)$ has an edge for every pair of edges of G having a common vertex. $K_{m,n}$ has m vertices of degree n and n vertices of degree m, thus the total number of edges of $L(K_{m,n})$ is

$$m \cdot \binom{n}{2} + n \cdot \binom{m}{2} = \frac{mn}{2}(m + n - 2)\,.$$

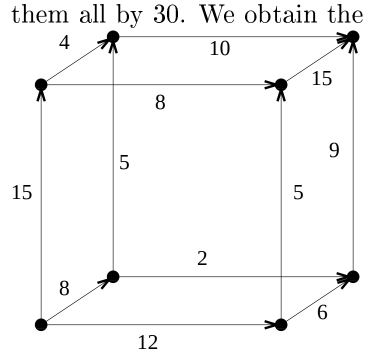
4. Find all the bipartite simple graphs G, such that $G \simeq \overline{G}$.
   Solution. Note that if one of the parts of G has 3 or more vertices then $\overline{G}$ has K_3 as a subgraph and can not thus be bipartite. Hence the parts of G have either 1 or 2 vertices and G consequently has 2, 3 or 4 vertices. On the other hand, since $G \simeq \overline{G}$, the graph K_n must have an even mumber of edges (where $n = |V(G)|$). Hence, $n = 4$ is the only possibility and G must have $\frac{6}{2} = 3$ edges. By considering two possible graphs with these parameters we conclude that the only option is P_4.

Test two

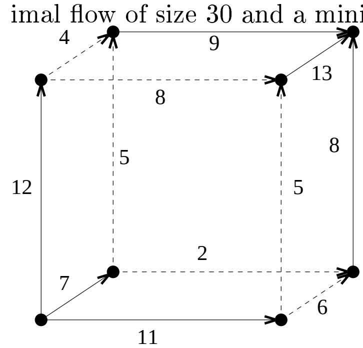
1. Find the maximal flow and a minimal cut in the following network:



   Solution. First convert all the arc capacities to integers by multiplying them all by 30. We obtain the following graph:



   Applying Ford-Fulkerson algorithm to this graph we find that it has maximal flow of size 30 and a minimal cut as follows:



   Hence the maximal flow in the original graph is $\frac{30}{30} = 1$ and the edges in the minimal cut are the same as in the last graph.

2. Prove that if all the edges of a graph G have pairwise different weigths, this graph has a unique spanning tree of minimal weigth.
   Solution. First note that the algorithm for finding a minimal spanning tree can in principle output every minimal spanning tree. Indeed, given a minimal spanning tree T and selecting its edges in the order of increasing weigth, we get T as an output.

The algorithm for finding a minimal spanning tree can only branch if at a certain step there are several edges of the same weigth. If all the edges of a graph G have different weigths, the output of the algorithm is uniquely determined, thus there can be only one minimal spanning tree.

3. Find the number of leaves in a tree with n vertices, where all the non-leaves have degree d.
   Answer: $\frac{nd-2n+2}{d-1}$.
   Solution. Let the number of leaves be x; then the tree has $n-x$ non-leaves. We know that the tree has $n-1$ edges, hence the sum of the vertex degrees is $2(n-1)$. On the other hand, there are x vertices of degree 1 and $n-x$ vertices of degree d, thus we get

$$\begin{aligned} 2(n-1) &= x \cdot 1 + (n-x) \cdot d \\ 2n-2-nd &= x - x \cdot d \\ \frac{nd-2n+2}{d-1} &= x \end{aligned}$$

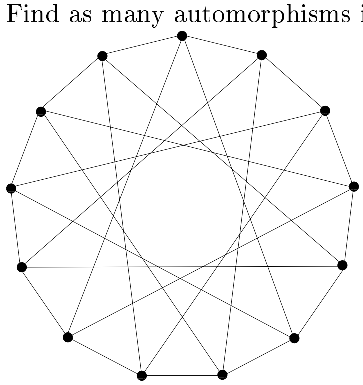
4. Two magicians show the following trick. One of them goes outside of the room, the second magician takes a deck of cards and gives it to the spectators. The spectators pick four cards out of the deck and show them to the second magician who removes one of the four cards. The spectators shuffle the three remaining cards, call the first magician back and show the remaining cards to him. The first magician guesses the card removed by his colleague. What is the greatest number of cards in the deck such that this trick is possible?
   Answer: 7.
   Solution. Let $\mathcal{D}$ denote the set of cards in the deck and let $|\mathcal{D}| = n$. Consider a bipartite graph with the parts $\mathcal{T}$ and $\mathcal{Q}$ consisting of all the unordered triples and quadruples of the set $\mathcal{D}$, respectively, and with edges between $T \in \mathcal{T}$ and $Q \in \mathcal{Q}$ iff $T \subset Q$. The second magician needs to "encode" all the possible quadruples with unique triples. He can do so only if the number of triples is at least as large as the number of quadruples. The inequality $|\mathcal{T}| \geq |\mathcal{Q}|$ gives us

$$\begin{aligned} \frac{n(n-1)(n-2)}{6} &\geq \frac{n(n-1)(n-2)(n-3)}{24} \\ 4 &\geq n-3 \\ 7 &\geq n \end{aligned}$$

   and hence the deck can have at most 7 cards. To prove that the graph has a suitable matching note that since $|\mathcal{D}| = 7$, we have $|\mathcal{T}| = |\mathcal{Q}| = 35$ and that the graph is regular with the degree of every vertex being 4. Thus this graph has a perfect matching which can be used to perform the trick.

Test three

1. Find as many automorphisms in the following graph as you can:



(*Remark.* Full marks are awarded if more than 50 automorphisms are found.)
Answer: 52.
Solution. We have 13 rotations and 13 rotations with reflection. Besides these, there is also an automorphist taking the outer cycle to inner and vice versa, doubling the overall number of automorphisms.

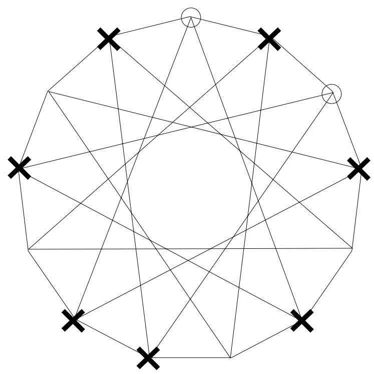
2. Prove that $r(3, 5) = 14$.
(*Hint.* You may find it useful to look at the first problem.)
Solution. First we know that

$$r(3, 5) \leq r(2, 5) + r(3, 4) = 5 + 9 = 14\,.$$

It remains to prove that there exists a graph on 13 nodes that has no K_3 nor O_5 as induced subgraphs. A suitable graph G has a vertex set $V(G) = \mathbb{Z}_{13}$ and edge set

$$E(G) = \{(a, b) : a - b \equiv 1 \bmod 13 \vee a - b \equiv 5 \bmod 13\}\,.$$

Absence of induced K_3 can be verified directly (e.g. by noting that three numbers, each of which is equal to either 1 or 5, can not sum up to a multiple of 13). In order to prove absence of induced O_5 note first that if we select five elements a, b, c, d, e from $\mathbb{Z}_{13}$, there must be two of them different from each other by at most 2. Since we can not have elements having difference 1, we must have two elements with difference exactly 2. Mark them on the graph and mark also the vertices prohibited by them:



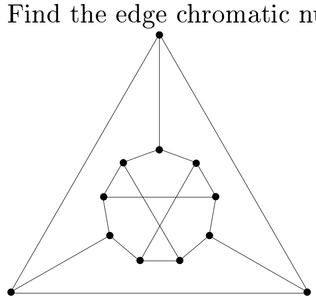
We see that the remaining three vertices must be chosen among four remaining ones, but this is impossible, since they are pairwise connected by an edge.

3. Let G be a connected palanar graph such that the degree of each vertex is at least 3 and that has at most 10 faces. Prove that G has at most 16 vertices.
Solution. Since each vertex has at least 3 neighbours, by double counting we obtain the inequality $m \geq \frac{3}{2}n$. Then substituting this inequality and the inequality $f \leq 10$ to the Euler formula, we get

$$10 \geq f = m - n + 2 \geq \frac{3}{2}n - n + 2 = \frac{1}{2}n + 2\,,$$

which implies $n \leq 16$.

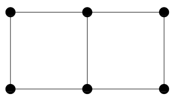
4. Find the edge chromatic number of the following Tietze graph:



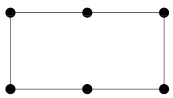
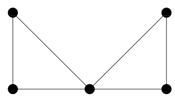
Answer: 4.
Solution. According to Vizing' theorem, $\chi'(G)$ may be either $\Delta(G)$ or $\Delta(G) + 1$. Tiete graph has $\Delta(G) = 3$, so the possibilities for $\chi'(G)$ are 3 and 4. Trying out the possibilities to color it with 3 colors we see that they all lead to a contradiction, so we must have $\chi'(G) = 4$.

Test four

1. Find the chromatic polynomial of the following graph:

   Solution. The chromatic polynomial of the given graph is a difference of the chromatic polynomials of the following graphs:

   The first one of them is C_6 with chromatic polynomial

   $$(k-1)^6 + (-1)^6(k-1) = k^6 - 6k^5 + 15k^4 - 20k^3 + 15k^2 - 5k$$

   and the chromatic polynomial of the second graph may be directly computed to be equal to

   $$k(k-1)^2(k-2)^2 = k^5 - 6k^4 + 13k^3 - 12k^2 + 4k\,.$$

   All in all, the required chromatic polynomial is

   $$k^6 - 7k^5 + 21k^4 - 33k^3 + 27k^2 - 9k\,.$$

2. Prove that all the simple graphs with at most 8 edges are planar.
   Solution. Kuratowski theorem states that a graph is non-planar iff it has a subgraph homeomorphic to K_5 or $K_{3,3}$. Note that K_5 has 10 edges and $K_{3,3}$ has 9 edges. Thus a graph with at most 8 edges can have none of these graphs as a homeomorphic subrgaphs (since turning this subrgaph into K_5 or $K_{3,3}$ may only lose edges) and hence it must be planar.

3. In a simple graph G call a vertex subset D *dominating*, if for every node $x \in V(G) \setminus D$ there exists a node $y \in D$ that is connected to x. Let $\Delta(G)$ denote the largest vertex degree of the graph and let $\delta(G)$ denote the size of the smallest dominating set. Prove the inequality

   $$\delta(G) \leq |V(G)| - \Delta(G)\,.$$

   Solution. Let a be a vertex with maximal degree, and $N(a)$ be the set of the $\Delta(G)$ vertices connected to a by an edge. We claim that the set $D = V(G) \setminus N(a)$ is dominating. Indeed, the only vertices not belonging to D are the ones from $N(a)$ and they are all connected to the vertex $a \in D$. Thus there is a dominating set of size $|V(G)| - \Delta(G)$, hence $\delta(G) \leq |V(G)| - \Delta(G)$, proving the required claim.

4. Consider a connected non-complete simple graph with n vertices. Prove that if the smallest dominating vertex subset of this graph has size k, then the vertices of this graph can be colored in $n-k$ colors so that no two

vertices of the same color are joined by an edge.

Solution. We need to prove that $\chi(G) \leq n - \delta(G)$. The conditions of the problem say that G is connected, but not a complete graph; hence it has no clique of size $\Delta(G) + 1$. If $\Delta(G) \geq 3$, we can use Brooks theorem to conclude that $\chi(G) \leq \Delta(G)$, so it is enough to prove that $\Delta(G) \leq n - \delta(G)$. But this is the claim of the previous problem.

If $\Delta(G) \leq 2$, the graph G must be either a cycle or a path. The inequality $\chi(G) \leq \Delta(G)$ in most of the cases still holds, the only exceptions being P_2 and cycles of odd length. Since $P_2 \simeq K_2$ and $C_3 \simeq K_3$, the only case to consider is when $G \simeq C_n$, where $n \geq 5$ is an odd integer. We have $\delta(G) = \frac{n-1}{2}$ and $\chi(G) = 3$. Thus the required inequality becomes

$$3 \leq n - \frac{n-1}{2}$$

which is true, since $n \geq 5$.