Graphs – first test

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1. Which ones of the graphs $L(Q_n)$ $(n \geq 2)$ are Eulerian? Semi-Eulerian? Hamiltonian? Semi-Hamiltonian?
   Solution. Since Q_n is Hamiltonian for each n, line graphs of these graphs are Hamiltonian, too. Besides,
   $$\forall v \in V(Q_n)\, \deg(v) = n\,,$$
   thus
   $$\forall v \in V(L(Q_n))\, \deg(v) = 2n - 2\,.$$
   Since these graphs are obviously connected, they are all also Eulerian.

2. Prove that if a simple graph has n vertices and more than $\frac{n^2}{4}$ edges then it is not bipartite. Find all the bipartite simple graphs with n vertices and exactly $\frac{n^2}{4}$ edges.
   Solution. Let the graph $G = (V, E)$ have n vertices and be bipartite with partition $V = V_1 \cup V_2$ and let $|V_1| = k$; then $|V_2| = n - k$. There are at most $k \cdot (n - k)$ edges, so we have to prove the inequality
   $$k \cdot (n - k) \leq \frac{n^2}{4}\,.$$
   But this is equivalent to $n^2 - 4kn + 4k^2 \geq 0$ and $(n - 2k)^2 \geq 0$, which is true. The equality holds iff $n = 2k$ and all the possible edges are present, i.e. in the case of the graph $K_{k,k}$, where $k = \frac{n}{2}$.

3. Let $G = (V, E)$ be a simple graph. Prove that if there exists a vertex subset $V_1 \subseteq V$ such that $|V_1| > \frac{|V|}{2}$ and the subgraph induced by V_1 is a null graph, then G is not Hamiltonian.
   Solution. Let there be a Hamiltonian cycle in the graph G. Since no two vertices of V_1 are connected by an edge, these vertices can not occur next to each other in this cycle. Thus, there must be at least as many vertices in $V \setminus V_1$ as there are vertices in V_1, a contradiction with the conditions of the problem.

4. A *triangle* in the simple graph G is a triple $\Delta = \{u, v, w\} \subseteq V(G)$ such that $\{u, v\}$, $\{v, w\}$, $\{u, w\} \in E(G)$. *Triangle graph* $T(G)$ of graph G

has the vertex set

$$V(T(G)) = \{\Delta \mid \Delta \text{ is a triangle in graph } G\}$$

and edge set

$$E(T(G)) = \{\{\Delta, \Delta_2\} \mid |\Delta_1 \cap \Delta_2| = 2\}.$$

Prove that

(a) $T(K_4) \cong K_4$,
(b) $\overline{T(K_5)} \cong \text{Pet}$.

Solution. (a) There are four triangles in K_4 and each two of them share a common edge.
(b)