Graphs – second test

26.10.2007

1. Prove that for any alcane isomer C_nH_m the equality $m = 2n + 2$ holds. (The basic structure of the alcane isomer is a tree with nodes C (atoms of carbon) and H (atoms of hydrogen) with degrees 4 and 1, respectively.)
Solution. We use induction on n. For $n = 1$ we have one atom of carbon with degree 4, hence the number of hydrogen atoms must be $4 = 2 \cdot 1 + 2$. Now assume we have an isomer $C_{n+1}H_{m'}$. After removing all the H vertices, we are still left with a tree consisting of C verices. This tree must have a leaf. Remove this vertex (together with three neighbouring H vertices) from the original isomer $C_{n+1}H_{m'}$ and substitute it with a H vertex. We obtain an isomer C_nH_m and using the induction hypothesis, we get $m = 2n+2$. On the other hand, $m' = m-3+1 = m-2 = 2(n+1)+2$ which was required to complete the step of induction.

2. The *grid graph* $G_{m,n}$ is given by its vertex set $V(G_{m,n}) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ and edge set $E(G_{m,n}) = E_1 \cup E_2$, where

$$E_1 = \{\{(i,j),(i,j+1)\} | i = 0,1,\ldots,m;\ j = 0,1,\ldots,n-1\}$$

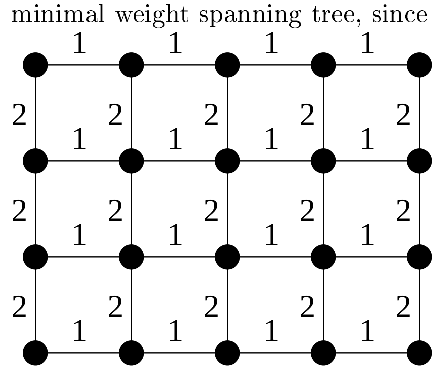
and

$$E_2 = \{\{(i,j),(i+1,j)\} | i = 0,1,\ldots,m-1;\ j = 0,1,\ldots,n\}.$$

The edges of $G_{m,n}$ are given weights by the rule

$$w(e) = \begin{cases} 1, & \text{if } e \in E_1 \\ 2, & \text{if } e \in E_2 \end{cases}.$$

Determine the weight of the minimal weight spanning tree of graph $G_{m,n}$. How many different minimal weight spanning trees are there (two spanning trees are considered different if their edge sets differ)?
Answer. The weight of the minimal spanning tree is $mn + 2m + n$ and there are $(n + 1)^m$ *such trees.*
Solution. First, it is clear that all the edges of E_1 must belong to every minimal weight spanning tree, since they do not form a cycle.



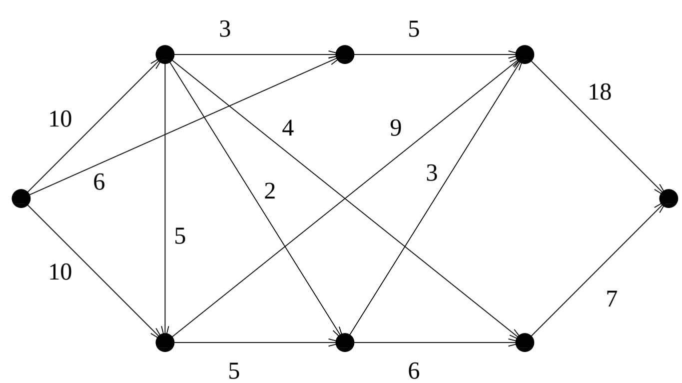
Next, from each set $\{\{(0,j),(1,j)\}|j = 0,1,\ldots,n\}$, $\{\{(1,j),(2,j)\}|j = 0,1,\ldots,n\}$, $\ldots$,$\{\{(m-1,j),(m,j)\}|j = 0,1,\ldots,n\}$ we must choose exactly one edge. Indeed, choosing no edge from some of these sets would give a non-connected graph and choosing more than one edge would create a cycle. Thus, the weight of the minimal spanning tree is

$$(m+1)\cdot n\cdot 1 + m\cdot 2 = mn + 2m + n\,.$$

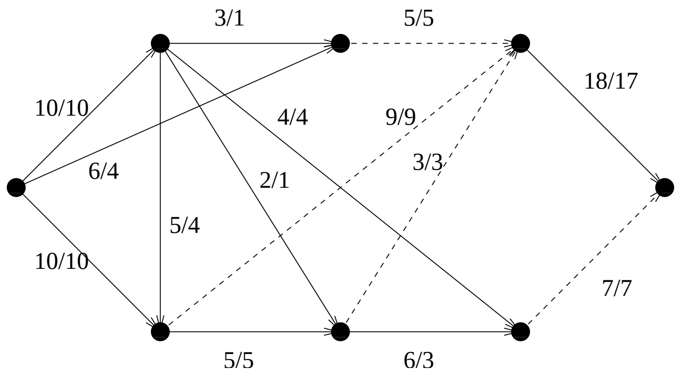
Altogether, there are $(n+1)^m$ possible choices of the last m edges, which is also the number of possible different spanning trees.

3. On the table, there are 36 coins of 6 different values (5, 10, 20 and 50 cents and 1 and 5 kroons), 6 coins each. The coins are randomly organized as a 6×6 array. Prove that it is possible to select one coin from each row so that all the values of the selected coins are different.
Solution. We will use Hall's theorem. First we create a bipartite graph with partition X, Y so that the elements of X are 6 rows of the array and the elements of Y are 6 possible values of the coins. We draw an edge $\{x,y\}$ if the row x has some coins of value y. Select a subset $S\subseteq X$ of rows. There are $6\times|S|$ coins in these rows, thus at least $|S|$ different values must be represented among them. Now applying the Hall's theorem we obtain a perfect matching in the given graph which proves the claim of the problem.

4. Find a maximal flow and a minimal cut in the following network.



Solution. The value of the maximal flow is 24; one possible such flow and a minimal cut are depicted in the figure below.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