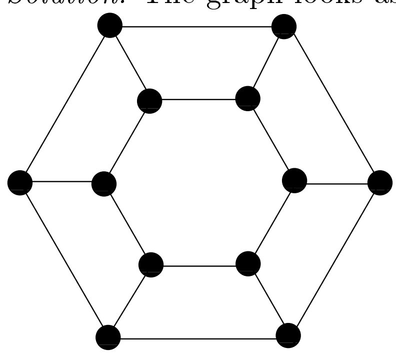
Graphs – third test

1. Let $n \geq 5$ be an integer. Consider a non-directed graph T_n having the vertex set $V(T_n) = \{1, 2, \dots, n\} \times \{1, 2\}$ and the edge set

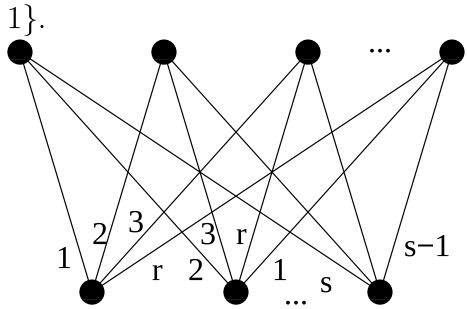
   $$E(T_n) = \{\{(a, b), (c, d)\} : (a = c+1 \,\&\, b = d) \vee (a = c \,\&\, b = 1 \,\&\, d = 2)\},$$

   where we take $n + 1 = 1$. Find $|Aut(T_n)|$.
   Answer: $4n$.
   Solution. The graph looks as follows:

   Consider any of its vertices. Two of its incident edges belong to a cycle of length n and one does not (note that $n > 4$). Thus, after moving a vertex to another by an automorphism (we have $2n$ ways of choosing the another vertex) we still can chooseone of the two ways of fixing its incident edges. Hence, altogether we have $2 \cdot 2n = 4n$ automorphisms.

2. Prove that $\chi'(K_{r,s}) = \max(r, s)$ by constructing an explicit edge coloring.
   Solution. Assume that $r \geq s$. Draw $K_{r,s}$ the way shown in the figure so that r vertices are above and s vertices below. Now successively colour the edges using colours $\{1, 2, \dots, r\}, \{2, 3, \dots, r, 1\}, \dots, \{s, \dots, r, 1, \dots, s-1\}$.

3. Prove that $r(4,5) \leq 32$.

Solution. We will use Ramsey theorem:

$$\begin{aligned} r(4,5) &\leq r(3,5) + r(4,4) \leq \\ &\leq r(2,5) + r(3,4) + r(3,4) + r(4,3) = \\ &= r(2,5) + 3 \cdot r(3,4) = 5 + 3 \cdot 9 = 32, \end{aligned}$$

since $r(2, 5 = 5)$ and $r(3,4) = 9$.

4. Prove that every planar connected 3-regular graph whose every face has at least 5 edges, has at least 20 vertices altogether.

Solution. Count the number of pairs (vertex,face) where a given vertex belongs to a given face. On one hand, since every face has at least 5 vertices, this number is at least $5f$. On the other hand, since every vertex has 3 incident edges, it also belongs to 3 faces. Hence the number of such pairs is $3n$. So we get the inequality $3n \geq 5f$.

If we count the number of pairs (vertex, edge) where a given vertex belongs to a given edge, we get the equality $3n = 2m$. Substituting these results to the Euler's formula, we get

$$\begin{aligned} 3n &\geq 5f \\ 3n &\geq 5(2 - n + m) \\ 3n &\geq 5(2 - n + \frac{3}{2}n) \\ 3n &\geq 10 + \frac{5}{2}n \\ \frac{1}{2}n &\geq 10 \\ n &\geq 20 \end{aligned}$$

which was required to prove.