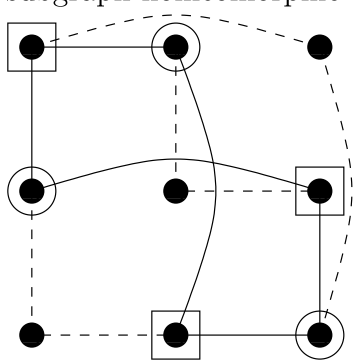
Graphs – fourth test

1. Let $n \geq 2$ be an integer. Determine which ones of the graphs R_n with vertex set $V(R_n) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and edge set

$$E(R_n) = \{\{(a,b),(c,d)\} : (a = c \,\&\, b \neq d) \vee (a \neq c \,\&\, b = d)\}$$

are planar.

Answer: R_2 is planar, all the graphs R_n $(n \geq 3)$ are non-planar.
Solution. It is easy to see that $R_2 \simeq C_4$ and is hence planar. The graph R_3 has a subgraph homeomorphic to $K_{3,3}$:



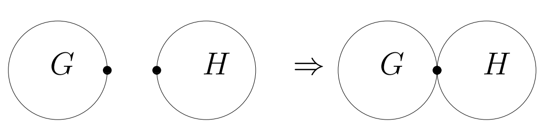
hence this graph is not planar. As obviously $R_3 \leq R_4 \leq \ldots$, all the graphs R_n $(n \geq 3)$ are non-planar.

2. Prove that for every simple graph G (where $|V(G)| \geq 1$) and integer $k > 0$, $P_G(k)$ is divisible by k.
Solution 1. We can use induction on $|E(G)|$. For the base case we have $|E(G)| = 0$ and $P_G(k) = k^n \vdots k$. For the induction step we can choose any edge e and use the formula $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$. By induction hypothesis, both $P_{G-e}(k)$ and $P_{G/e}(k)$ are divisible by k, so must be $P_G(k)$ as well.
Solution 2. Let C be the set of all the colorings of graph G with k colors from the set $\{1, 2, \ldots, k\}$. We will show how to partition the set C into k pairwise non-interseting parts with the same size. Take any $v \in V(G)$ and consider the following sets of colorings:

$$C_i = \{c \in C \,:\, c(v) = i\}, \quad i = 1, 2, \ldots, k\,.$$

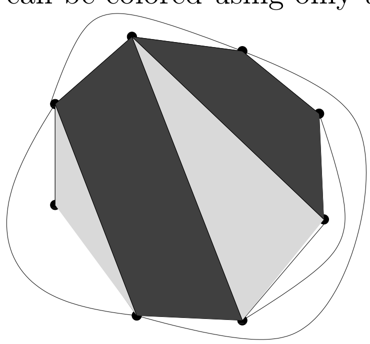
It is obvious that $\cup_{i=1}^k C_i = C$ and that the sets C_i are pairwise non-intersecting. They are also of the same size, since any permutation of colors turns a correct coloring into a correct coloring once again. Thus starting the coloring from vertex v, no color has any preference over the others and thus the number of colorings with the color of v being fixed to i is the same for all i.

3. Given two simple graphs G and H we form a *glued* graph $\Gamma(G, H)$ by joining them in one vertex:



Find the expression for the chromatic polynomial of the glued graph $\Gamma(G, H)$ in terms of the chromatic polynomials of the original graphs G and H .
Answer: $P_{\Gamma(G,H)}(k) = \frac{P_G(k) \cdot P_H(k)}{k}$.
Solution. First color the graph G; there are $P_G(k)$ ways to do it with k colors. Next we start coloring H from the vertex common to the copies of G and H. Since this vertex is already colored, we have $\frac{P_H(k)}{k}$ ways to color the rest of H as was proven in the previous problem.

4. We say that the coloring of the faces of a planar graph is *correct* if every two faces having a common edge are colored differently. Prove that the faces of a Hamiltonian planar graph can be correctly colored with four colors (without using the general Four Color Theorem by Appel and Haken).
Solution. Consider the Hamiltonian cycle in a given planar Hamiltonian graph G. Then all the edges not belonging to the cycle must still join two vertices on the cycle. There are two kinds of such edges in the planar drawing of G – the ones inside and the ones outside the cycle. It is clear that the faces inside the cycle can be colored using only two colors:



Silimar claim also holds true for the edges and faces outside the cycle (to see this, consider such transformation of the drawing of the graph that excahnges the faces inside and outside the cycle). Hence four colors are sufficient to color such a graph.