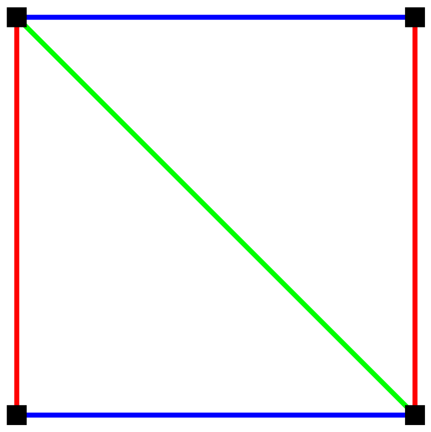
Coloring edges

Let $G = (V, E)$ be a graph without loops. Its *(correct) edge coloring with k colors* is a function $\gamma : E \longrightarrow \{1, \ldots, k\}$ such that

- for any two different edges $e_1, e_2 \in E$ with a common endpoint we have $\gamma(e_1) \neq \gamma(e_2)$.

Stating it otherwise, all the edges incident with some vertex must be colored differently.



Example: let us consoder a set of (school) classes X and a set of teachers Y. For each class it is known how many lessons a given teacher must teach to this class.

The task is to compose a time-table for the school.

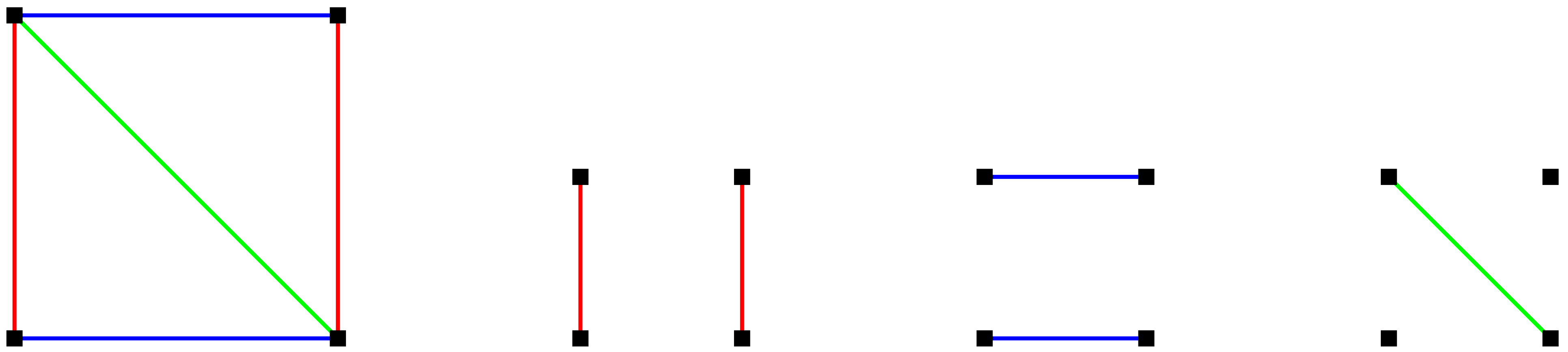
Consider a bipartite graph with vertex set $X \cup Y$ and the number of edges between $x \in X$ and $y \in Y$ showing, how many lessons the teacher y teaches to the class x.

The time-table can be represented as a correct edge coloring, where the edge colors are possible time slots for the lessons.

Let $G = (V, E)$ be a graph, γ its edge coloring and i one of the colors.

The set $\{e \mid e \in E, \gamma(e) = i\}$ is a matching.

Coloring the edge set can be thought of as partitioning this set into matchings.



Let $G = (V, E)$ be a graph. Assume it has a correct edge coloring with k colors, but no correct edge coloring with $k - 1$ colors.

The number k is called *edge chromatic number* and denoted $\chi'(G)$.

Let $\Delta(G) = \max_{v \in V} \deg(v)$ be the maxmal vertex degree of graph G.

Obviously, $\chi'(G) \geq \Delta(G)$.

An example where $\chi'(G) > \Delta(G)$: odd cycles.

Theorem. In a bipartite graph G we have $\chi'(G) = \Delta(G)$.

Proof. First turn G into a $\Delta(G)$-regular graph.

1. If one of the vertex set parts has less vertices than the other, then add the missing number of vertices to make the parts equal.

2. If some of the vertices in one part has a degree less than $\Delta(G)$, then a similar vertex must also exist in the other part. Join these two by an edge.

If the edges of the new graph can be colored using $\Delta(G)$ colors, the same holds true for the original graph as well.

Thus we can consider only k-regular bipartite graphs G.

1. k-regular bipartite graph G has a complete matching M_1.
2. Remove the edges of M_1. The remaining graph is a $(k-1)$-regular bipartite graph.
3. This graph has a complete matching M_2.
4. Remove the edges of M_2. The remaining graph is a $(k-2)$-regular bipartite graph.
5. etc.

This way we partition the edge set of G into k perfect matchings $M_1, \ldots, M_k$. These matchings give a suitable coloring. $\square$

Theorem (Vizing). Let $G = (V, E)$ be a simple graph. Then $\chi'(G) \leq \Delta(G) + 1$.

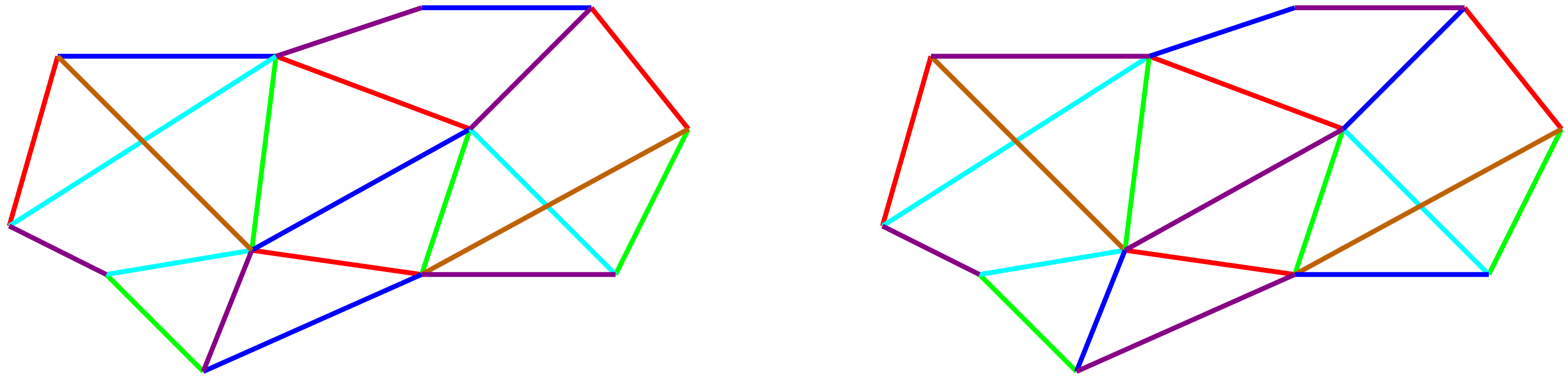
Proof is by induction over the number of vertices. The claim is obvious if $|V| = 1$.

We have to show the following:

> Let $G = (V, E)$ be a simple graph and let $k = \Delta(G) + 1$. Choose a vertex $v \in V$ in graph G and let the edges of $G \backslash v$ be colorable with k colors. Then the edges of G can also be colored with k colors.

We will prove this statement using induction over k. In fact, we will even prove a slightly stronger result.

Lemma. Let $G = (V, E)$ be a graph and γ its edge coloring. Let $E' \subseteq E$ be an edge subset being colored with some two colors. Consider the graph $G' = (V, E')$.

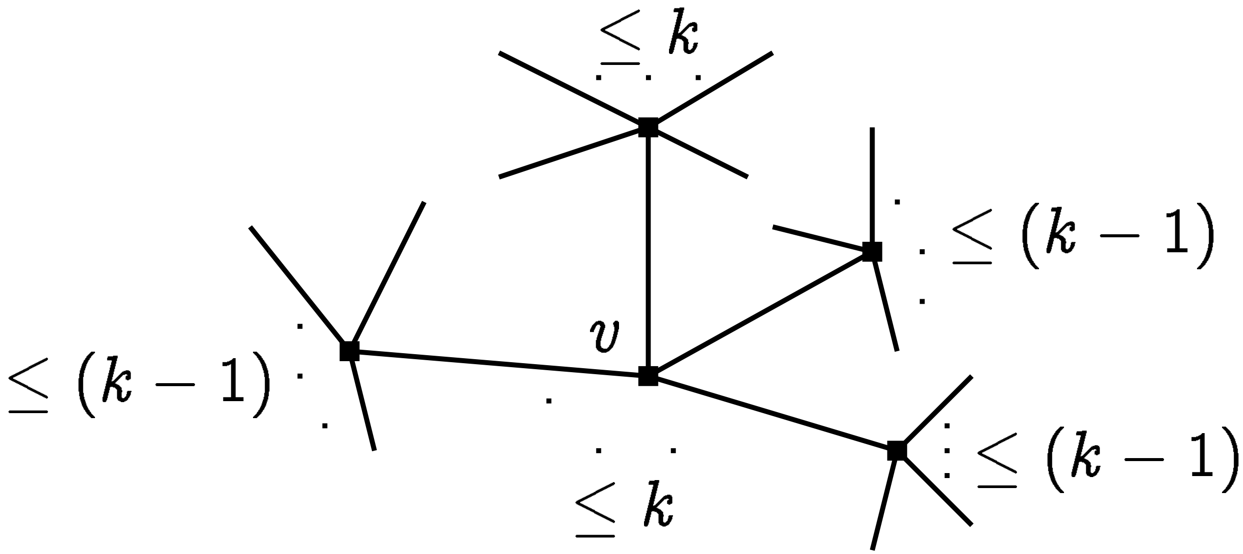


Let H be a connected component of graph G'. If we exchange the colors of the edges of H, we again get a correct coloring of the edges of G

Proof is pretty straightforward. $\square$

Lemma. Let $G = (V, E)$ be a simple graph and $k \in \mathbb{N}$. Let $v \in V$ be such that

- $\deg(v) \leq k$. If $w \in V$ is the neighbour of v, then $\deg(w) \leq k$.
- Vertex v has at most one neighbour with degree k.



Let the edges of $G \backslash v$ be colorable with k colors. Then the edges of k are colorable with k colors.

Proof by induction on k.

Base. $k = 1$.

Thus $\deg(v) = 0$ or $\deg(v) = 1$.

If $\deg(v) = 0$, the edges of G coincide with the edges of $G \backslash v$.

If $\deg(v) = 1$, then let u be the neighbour of v. According to the assumption of the Lemma we have $\deg(u) \leq 1$, thus $u — v$ is a connected component of G.

The coloring of G can be obtained from the coloring of $G \backslash v$ bu coloring the edge between u and v using the only available color.

Step. $k > 1$.

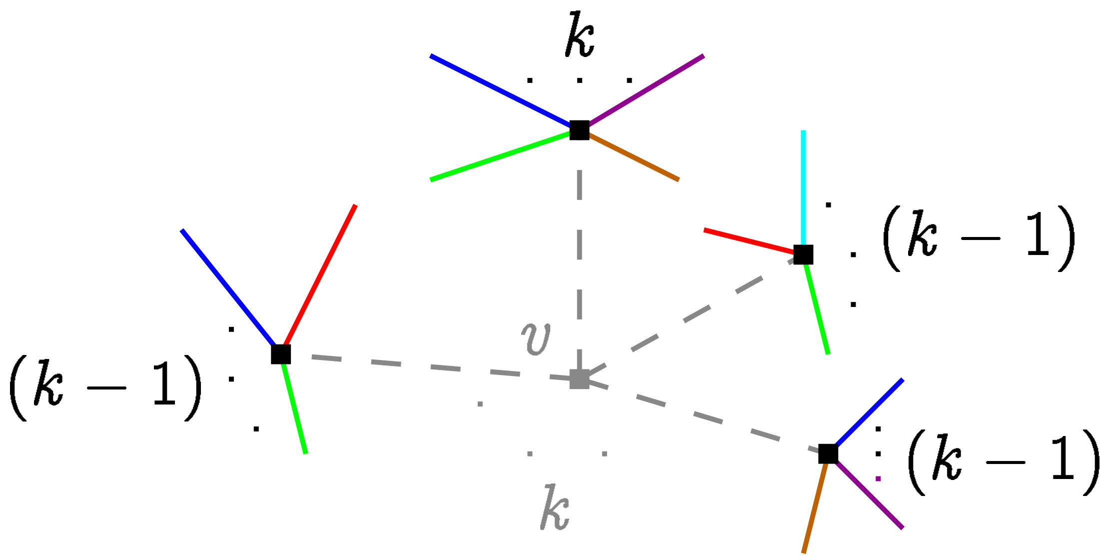
As long as $\deg(v) < k$, we add another vertex u and an edge u — v to the graph G.

As long as the degree of some neighbour v' of v is less than k or $k-1$, we add another vertex u and an edge u — v' to the graph G.

Thus we get a graph G with equalities holding in all the inequalities in the statement of the Lemma.

The modified graph is colorable with k colors iff the original graph was.

Let γ be the coloring of the graph $G \setminus v$ with k colors.



Consider the neighbours of the (removed) vertex v. For each $i \in \{1, \ldots, k\}$ let X_i be the set of such neighbours that have no incident edges colored with color i.

One of these vertices belongs to exactly one of the sets X_i, all the others belong to exactly two of these. Thus $\sum_{i=1}^{k} |X_i| = 2k - 1$.

We will be looking for γ such that there is a color i with $|X_i| = 1$.

That is, the edges colored with i are incident with all the neighbours of v, except for one.

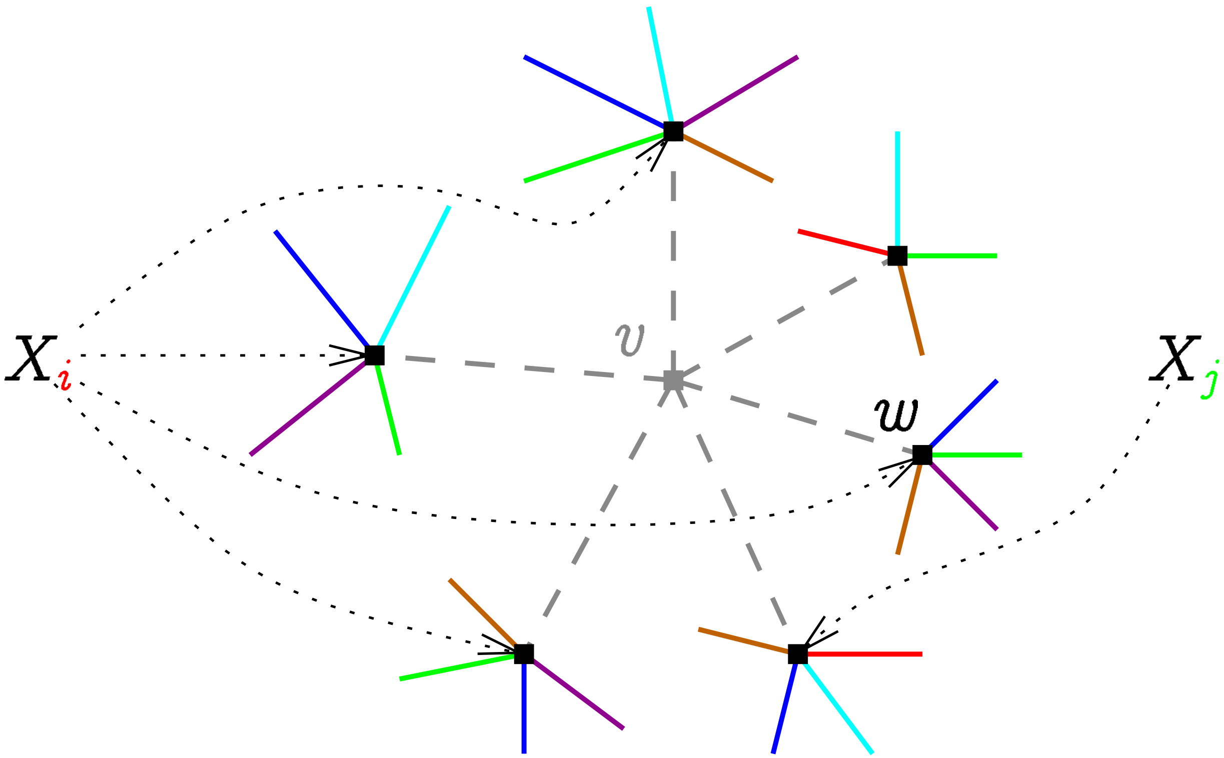
We will first show that we can choose γ so that for every $i, j \in \{1, \ldots, k\}$ we have $\big||X_i| - |X_j|\big| \leq 2$.

To do that we will prove that if for some i, j we have $|X_i| - |X_j| \geq 3$, then there is a coloring γ' such that $|X_i|$ is decreased by 1 and $|X_j|$ is increased by 1.

We will also prove that after a finite number of such steps $(\gamma \to \gamma')$ there will be no such i and j.

Let i and j be such that $|X_i| - |X_j| \geq 3$. Let $w \in X_i \setminus X_j$.

I.e., the number of vertices having an incident edge of color j is larger by at least 3 than the number of vertices having an incident edge of color i.

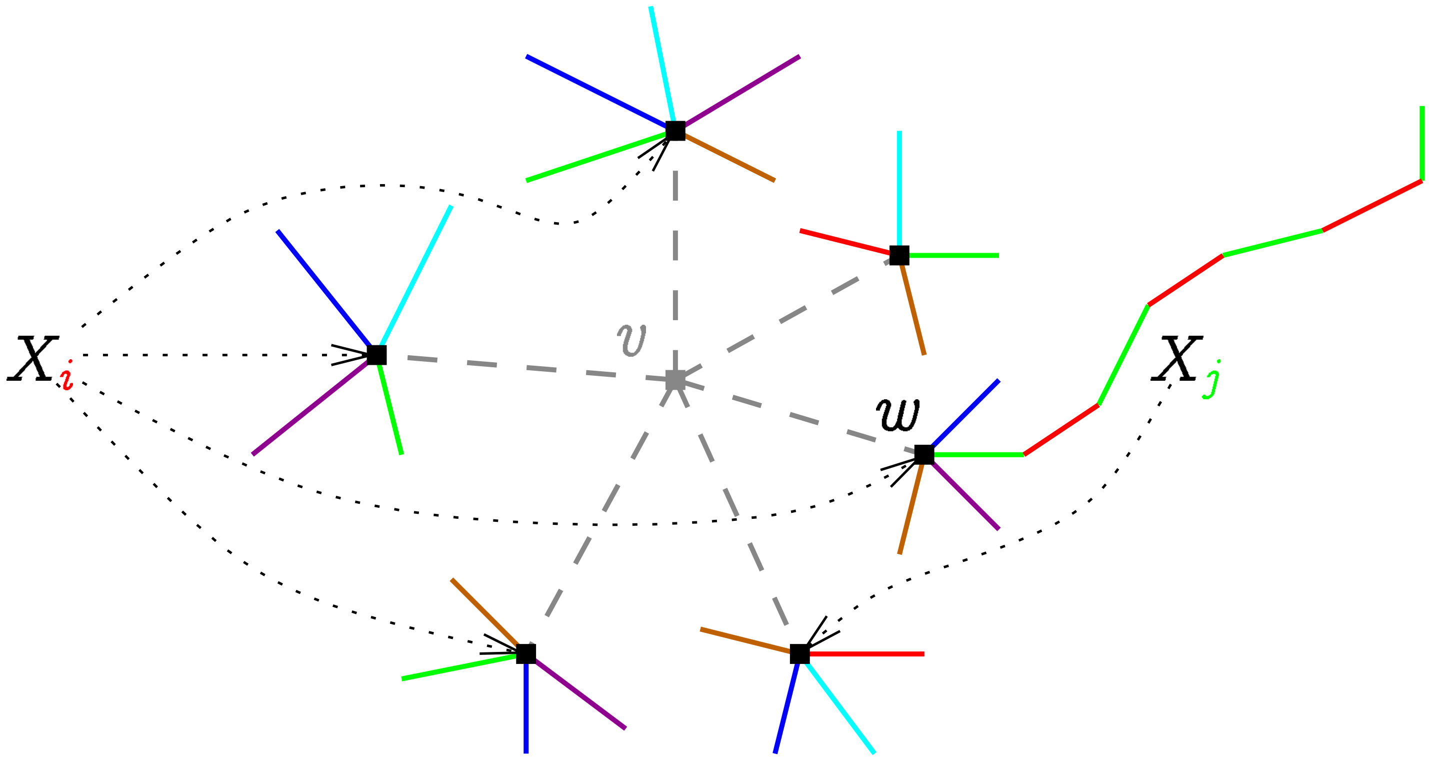


Let $E' \in E$ be the set of all edges e such that $\gamma(e) = i$ or $\gamma(e) = j$. Consider the graph $G' = (V, E')$.

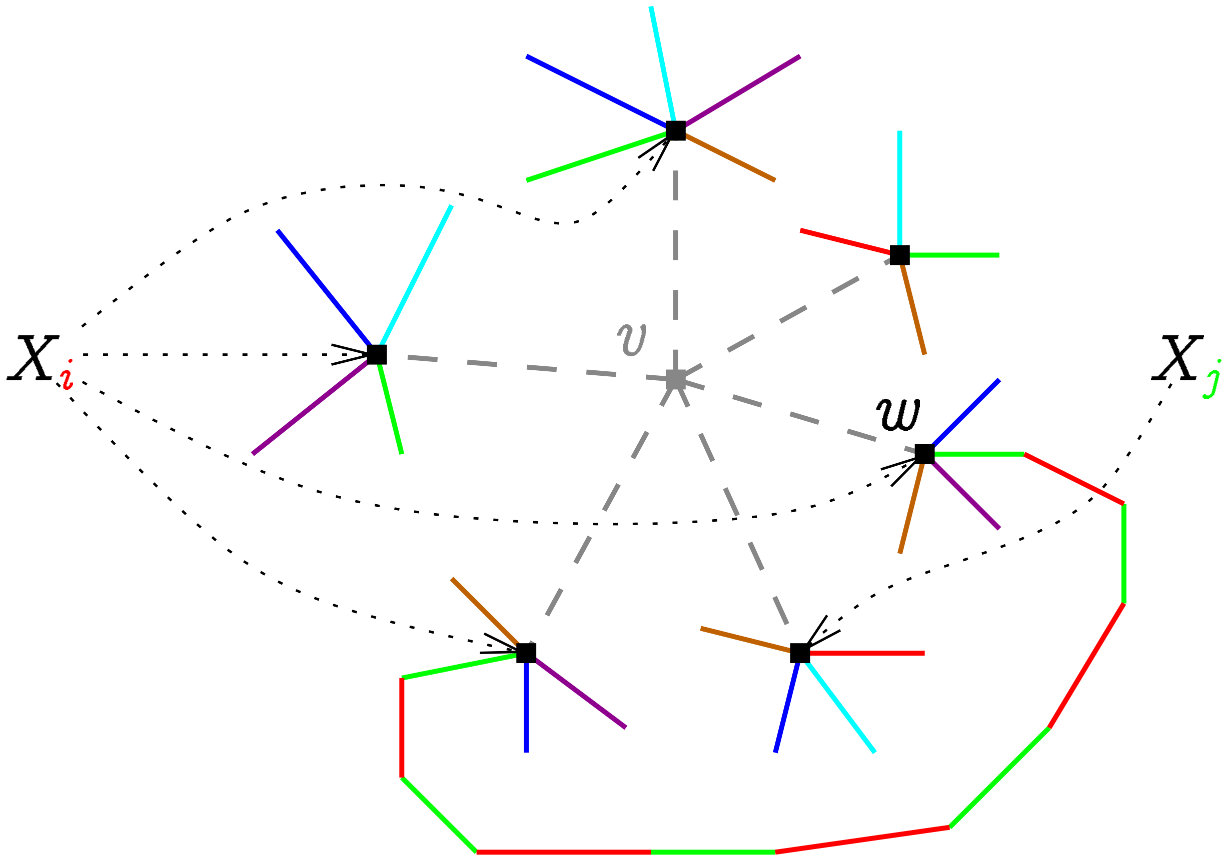
In G', all vertex degrees are ≤ 2. Thus the connected components of G' are isolated vertices, paths and cycles.

The vertex $w \in X_i \backslash X_j$ is an endpoint of some path.

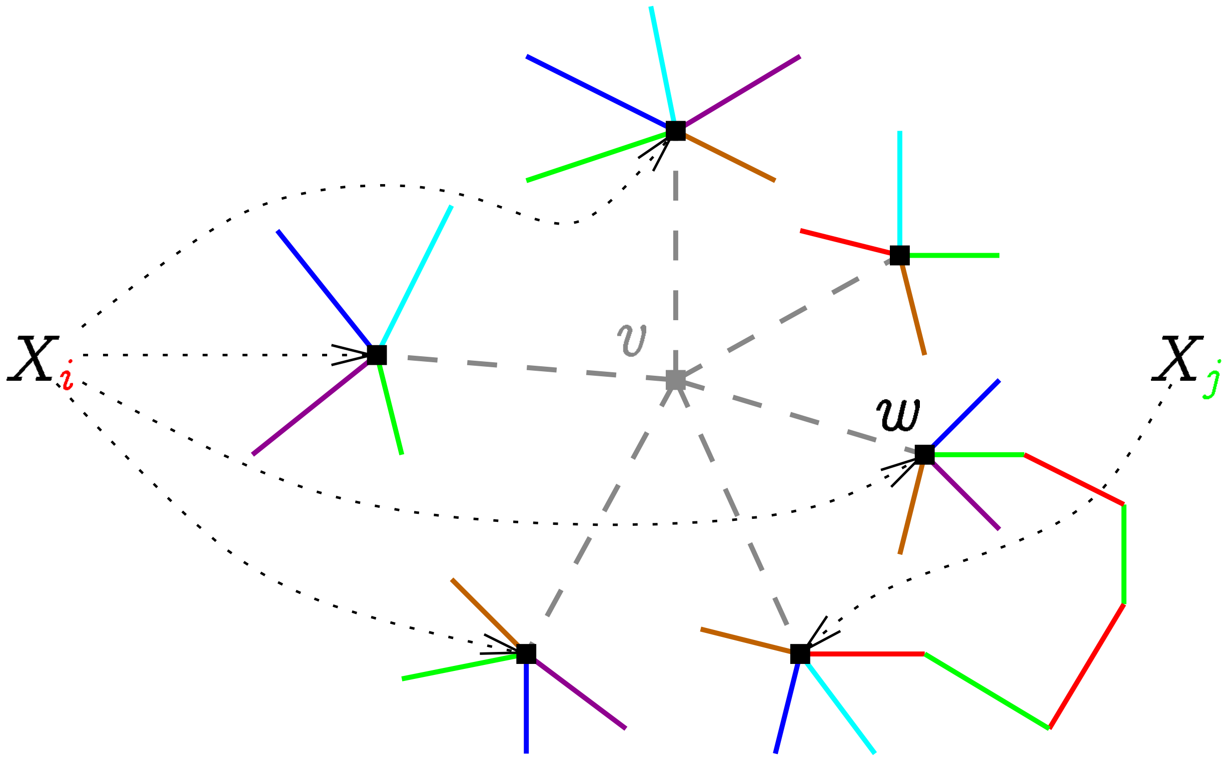
Where can the other endpoint be?



Somewhere else in graph G



In a vertex of the set $X_i \setminus X_j$

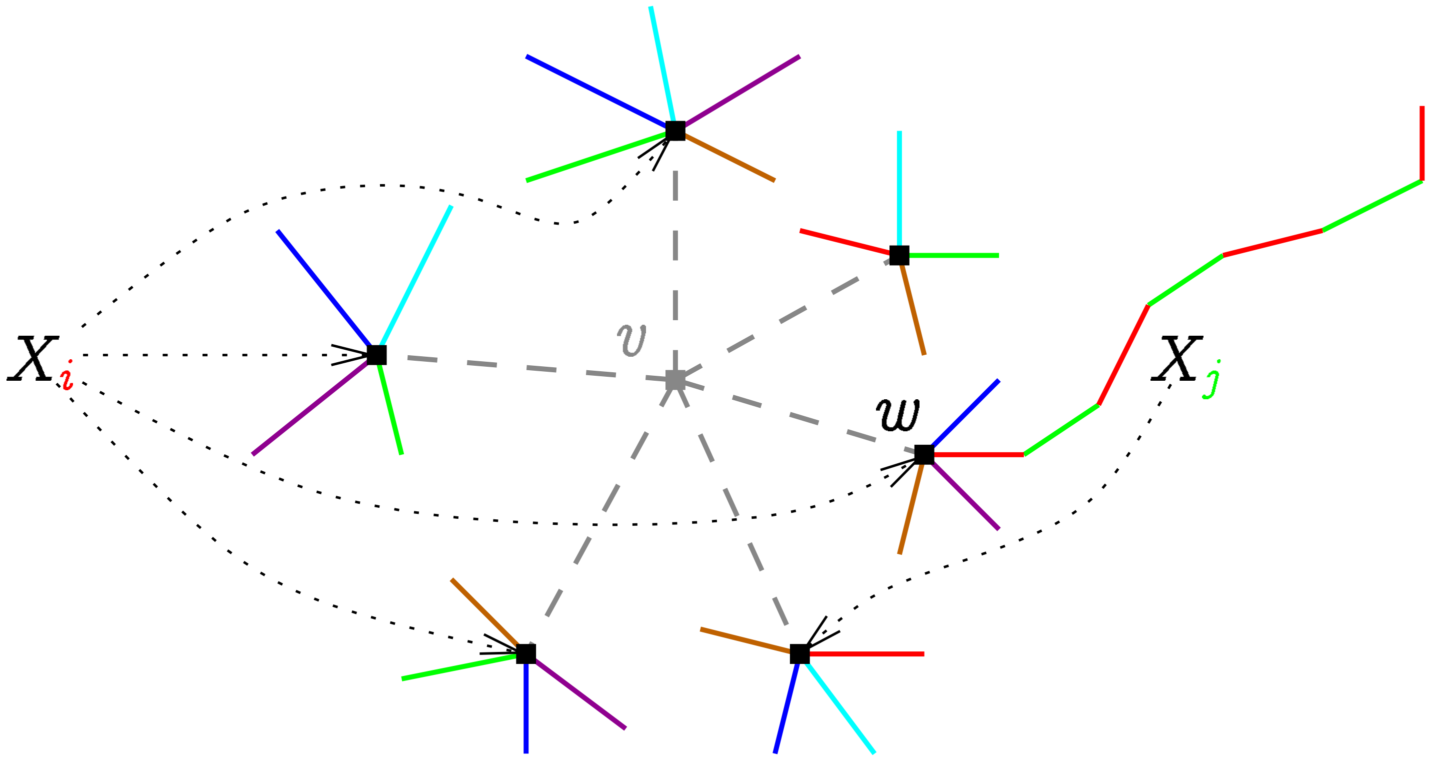


In a vertex of the set $X_j \backslash X_i$

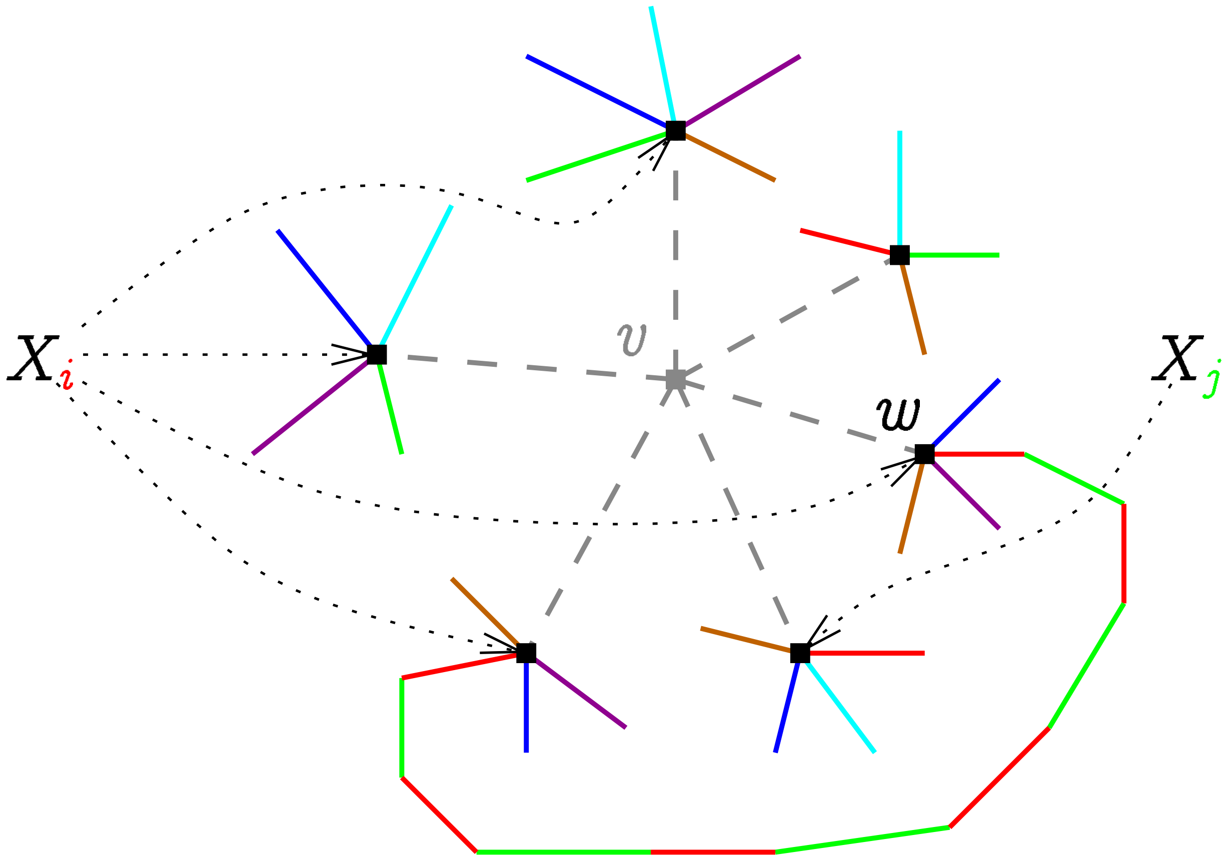
Since $|X_i| > |X_j|$, there exists $w \in X_i \backslash X_j$ such that the path that starts in it (being a connoected component in G') ends somewhere else than in the set $X_j \backslash X_i$.

In this path we exchange the colors i and j. We get a new coloring γ'.

$|X_i|$ and $|X_j|$ will change as follows:



$|X_i|$ decreases by one, $|X_j|$ increases by one



$|X_i|$ decreases by two, $|X_j|$ increases by two

To show the finiteness of the process, we need a *monovariant*, i.e. a quantity describing a coloring γ of the graph $G\backslash v$, such that

- On each step $(\gamma \to \gamma')$ it changes by a positive integer in a certain direction (e.g. decreases strictly).
- It has a fixed bound in this direction (e.g. 0).

A suitable quantity is $\sum_{i=1}^{k} |X_i|^2$.

Indeed, let $n_i, n_j \in \mathbb{N}$ such that $n_i - n_j \geq 3$. Then

$$(n_i-1)^2+(n_j+1)^2 = {n_i}^2+{n_j}^2-2(n_i-n_j)+2 \leq {n_i}^2+{n_j}^2-4$$
$$(n_i-2)^2+(n_j+2)^2 = {n_i}^2+{n_j}^2-4(n_i-n_j)+8 \leq {n_i}^2+{n_j}^2-4$$

We have shown that there is a coloring γ, such that the cardinalities of X_i differ by at most 2.

Average cardinality of the sets X_i is a bit less than 2 (namely $\frac{2k-1}{k}$). Thus the possible sets of cardinalities of X_i are $\{0, 1, 2\}$ and $\{1, 2, 3\}$.

If we have $\{1, 2, 3\}$, then there must exist i such that $|X_i| = 1$, otherwise the average cardinality is at least 2.

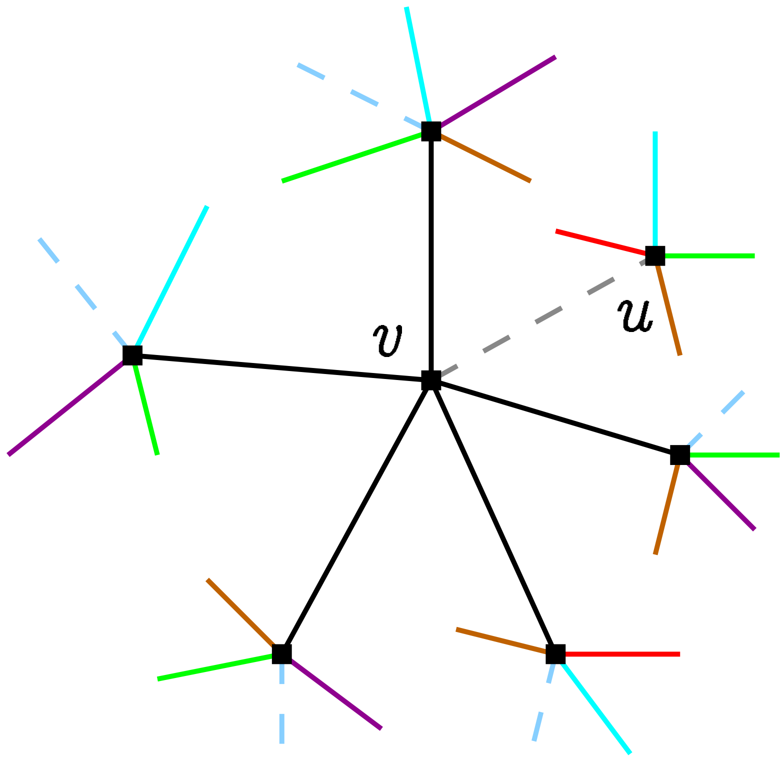
If we have $\{0, 1, 2\}$, then there must exist i such that $|X_i| = 1$, since the sum of cardinalities of X_i is odd $(2k - 1)$.

W.l.o.g. assume that this i is k. Let $\{u\} = X_k$.

Let H be obtained from G by deleting

- all edges that γ colors with color k;
- the edge between v and u.

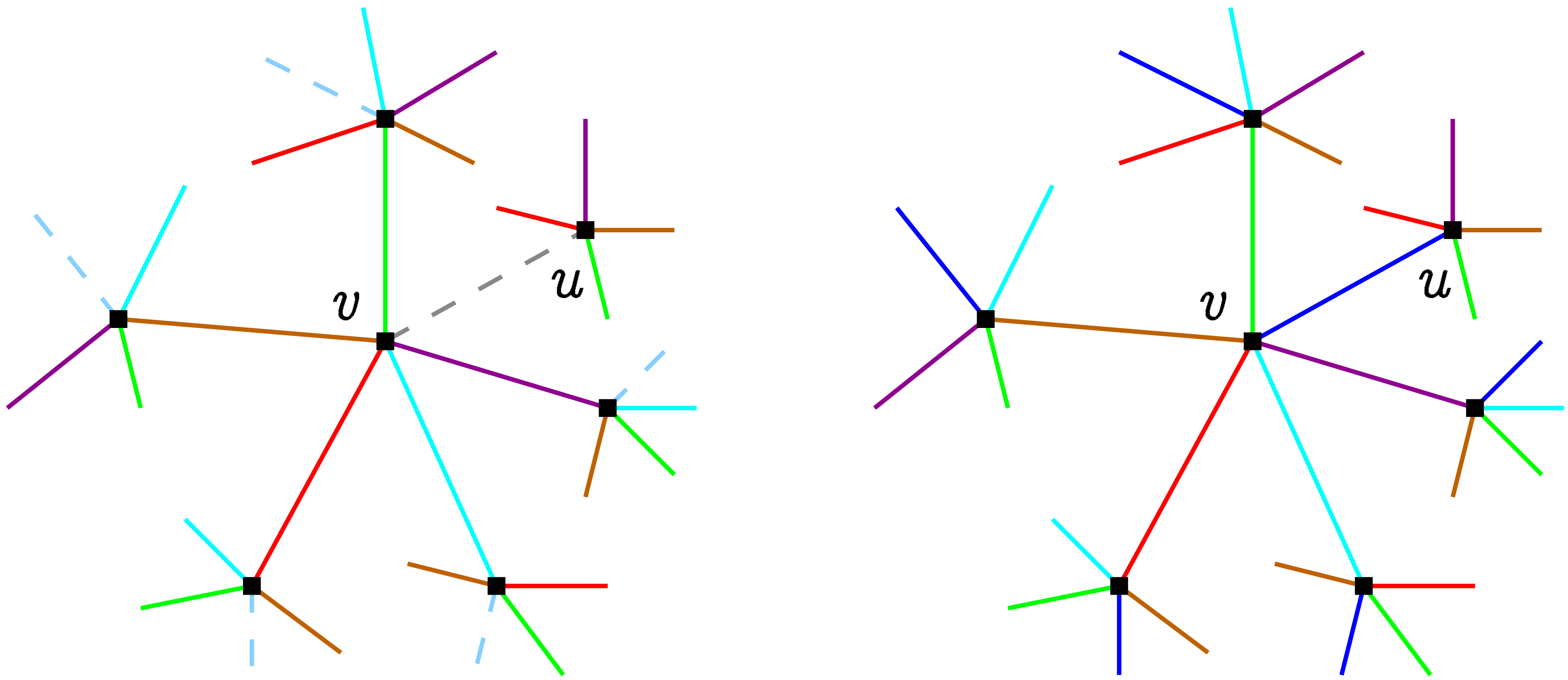
All the deleted edges form a matching in G.



Coloring γ without the color k is a coloring of the edges of $H \backslash v$ using $(k-1)$ colors.

The degree of v and its every neighbour (in H) has decreased by 1.

Induction hypothesis can be applied to graph H and vertex v. Thus the edges of H can be colored with $k-1$ colors. Let γ' be such a coloring.



We obtain the required coloring of G with k colors by coloring all the deleted edges with color k. □