Ramsey theory
Probabilistic proofs

Let $G = (V, E)$ be a graph. The vertex subset $S \subseteq V$ is called a *clique* if any two (different) vertices $u, v \in S$ are joined by an edge in G.

In other words, S is a clique if the induced subrgraph $G[S]$ is complete.

The vertex subset $S \subseteq V$ is called *independent* if none of the two vertices of S are joined by an edge.

In other words, S is independent if the induced subrgraph $G[S]$ is a null graph.

Proposition. Let $G = (V, E)$ be a simple graph such that $|V| \geq 6$. Then this graph has a 3-element clique or a 3-element independent vertex subset.

Proof. Let $v \in V$ be a vertex and let

- $X = N(v)$ (the set of neighbours of v);
- $Y = \overline{N}(v) = V \setminus (X \cup \{v\})$ (the non-neighbours of v).

Since $|X| + |Y| = |X \cup Y| = |V| - 1 \geq 5$, we have $|X| \geq 3$ or $|Y| \geq 3$. Assume $|X| \geq 3$. There are two options:

- X is an independent set.
- There exist $u, w \in X$, such that $(u, w) \in E$. Then $\{u, v, w\}$ is a clique.

The case $|Y| \geq 3$ is similar (instead of G we have $\overline{G}$). $\square$

Let $r(k,l)$ denote the least integer (if it exists) such that for each simple graph $G = (V, E)$, where $|V| \geq r(k,l)$,

$$K_k \hookrightarrow G \text{ or } O_l \hookrightarrow G \text{ holds.}$$

[Here $\hookrightarrow$ denotes being an induced subrgaph.]

We will show that $r(k,l)$ exists for all $k, l \in \mathbb{N}$ and we will also give some upper and lower bounds.

The first proposition showed that $r(3,3)$ exists and is at most 6.

Since $K_3 \not\hookrightarrow C_5$ and $O_3 \not\hookrightarrow C_5$, we have $r(3,3) = 6$.

Lemma. If $r(k,l)$ exists, then $r(l,k)$ also exists and $r(l,k) = r(k,l)$.

Proof. Obvious – we can exchange edges and non-edges. $\square$

Lemma. Let $k, l \in \mathbb{N}$. The quantities $r(k,1)$ and $r(k,2)$ exist. More precisely, $r(k,1) = 1$ and $r(k,2) = k$. Similarly, $r(1,l) = 1$ and $r(2,l) = l$.

Proof. O_1 is just a single vertex which is contained in any other graph. Thus $r(k,1) = 1$.

Let $G = (V, E)$ be a simple graph with $|V| = k$. If $G = K_k$ then $K_k \hookrightarrow G$. If $G \neq K_k$ then consider $u, v \in V$ such that $(u, v) \notin E$. Then $G[\{u, v\}] = O_2$.

We have shown that $r(k,2) \leq k$. At the same time $K_k \not\hookrightarrow K_{k-1}$ and $O_2 \not\hookrightarrow K_{k-1}$. Thus $r(k,2) = k$. $\square$

Theorem. Let $k, l \in \mathbb{N}$, such that $k \geq 2$ and $l \geq 2$. Then $r(k, l)$ exists and $r(k, l) \leq r(k-1, l) + r(k, l-1)$.

Proof. Induction over $k + l$.

Base. $k + l = 4$. Then $k = l = 2$. The previous lemma gives

$$r(2,2) = 2 = 1 + 1 = r(1,2) + r(2,1) \quad .$$

Step. Induction hypothesis gives that $r(k-1, l)$ and $r(k, l-1)$ exist.

Let $G = (V, E)$ be a simple graph, such that $|V| = r(k-1, l) + r(k, l-1)$.

Let $v \in V$; consider the sets $N(v)$ and $\overline{N}(v)$.

Since $|N(v)| + |\overline{N}(v)| = r(k-1,l) + r(k,l-1) - 1$, at least one of the following inequalities holds:

1. $|N(v)| \geq r(k-1,l)$.
2. $|\overline{N}(v)| \geq r(k,l-1)$.

In the first case consider the graph $G[N(v)]$. There are two options:

- $K_{k-1} \hookrightarrow G[N(v)]$. Let $S \subseteq N(v)$ be a $(k-1)$-element clique. Then $S \cup \{v\}$ is a k-element clique.
- $O_l \hookrightarrow G[N(v)]$. Then $O_l \hookrightarrow G$, too.

In the other case consider the graph $G[\overline{N}(v)]$. There are two options:

- $O_{kl-1} \hookrightarrow G[\overline{N}(v)]$. Let $S \subseteq \overline{N}(v)$ be an $(l-1)$- element independent set. Then $S \cup \{v\}$ is an l-element independent set.
- $K_k \hookrightarrow G[\overline{N}(v)]$. Then $K_k \hookrightarrow G$, too.

We have shown that any $(r(k-1,l)+r(k,l-1))$-vertex graph has a k-element clique or an l-element independent set. Thus $r(k,l)$ is at most $r(k-1,l)+r(k,l-1)$. $\square$

Corollary. If $r(k-1,l)$ and $r(k,l-1)$ are even, then $r(k,l) \le r(k-1,l) + r(k,l-1) - 1$.

Proof. Let $G = (V,E)$ be a simple graph, where $|V| = r(k-1,l) + r(k,l-1) - 1$. Let $v \in V$ be such that $|N(v)|$ is even. Such a v exists, because $|V|$ is odd.

Since both $|N(v)|$ and $|\overline{N}(v)|$ are even, at least one of the following inequalities holds:

1. $|N(v)| \ge r(k-1,l)$.
2. $|\overline{N}(v)| \ge r(k,l-1)$.

The proof can be completed the same way as the proof of the previous theorem. $\square$

Proposition. $r(k,l) \leq \binom{k+l-2}{k-1}$.

Proof. $r(1,1) = r(1,2) = r(2,1) = 1 = \binom{0}{0} = \binom{1}{0} = \binom{1}{1}$.

We use induction over $k+l$ for the other values of k and l. We have completed the base $k+l \leq 3$.

Step. Let $k+l \geq 4$. Then $r(k,l) \leq r(k-1,l)+r(k,l-1) \leq \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1} = \binom{k+l-2}{k-1}$. $\square$

The numbers $r(k, l)$ can be generalized.

$r(k, l)$ is the least number n, such that if the edges of K_n are colored with two colors (not necessarily in a correct manner) then there exists a monochromatic subrgaph K_k of the first color or a monochromatic subrgaph K_l of the second color.

Let $r(a_1, \ldots, a_k)$ be the least number n, such that if the edges of K_n are colored with k colors, then there exists a monochromatic subrgaph K_{a_i} of the color a_i.

The inequality

$$r(a_1, \ldots, a_k) \leq$$
$$r(a_1 - 1, a_2, \ldots, a_k) + r(a_1, a_2 - 1, a_3, \ldots, a_k) + \cdots +$$
$$r(a_1, \ldots, a_{k-1}, a_k - 1) - (k-2)$$

holds and $r(\ldots, 1, \ldots) = 1$.

Proof is similar to the case $k = 2$.

Theorem. If $k \geq 2$, then $r(k, k) \geq 2^{k/2}$.

Proof. Let $n < 2^{k/2}$ and let $\mathbf{G}_n$ be the set of all n-vertex simple graphs. We have to show that there exists $G \in \mathbf{G}_n$, such that $K_k \not\hookrightarrow G$ and $O_k \not\hookrightarrow G$.

Consider a set $\mathcal{X}$ and some predicate P on it, i.e. a function $P : \mathcal{X} \to \{\text{true}, \text{false}\}$. Say, we need to prove that there exists $x \in \mathcal{X}$, such that $P(x)$ holds.

For that it is enough to prove that selecting a *random* element $x \in \mathcal{X}$, we have $\mathbf{P}[P(x)] > 0$.

In order to define what it means to select a graph randomly from the set $\mathbf{G}_n$, we need to fix a probability distribution on this set.

Consider the elements of $\mathbf{G}_n$ to be *labeled* simple graphs on n vertices (with vertex labels from the set $\{1, \ldots, n\}$). Then $|\mathbf{G}_n| = 2^{\binom{n}{2}}$.

Let the vertex set of $G \in \mathbf{G}_n$ be $\{v_1, \ldots, v_n\}$, where the label of v_i is i.

Let G be a uniformly chosen random labeled graph from the set $\mathbf{G}_n$

We will find upper bounds for $\mathbf{P}[K_k \hookrightarrow G]$ and $\mathbf{P}[O_k \hookrightarrow G]$.

$$\mathbf{P}[K_k \hookrightarrow G] =$$

$$\frac{\text{the number of graphs in } \mathbf{G}_n \text{ containing } k\text{-element clique}}{|\mathbf{G}_n|} \leq$$

$$\frac{1}{|\mathbf{G}_n|} \cdot \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left| \{G \in \mathbf{G}_n \mid G[\{v_{i_1}, \ldots, v_{i_k}\}] \cong K_k\} \right| =$$

$$\frac{1}{2^{\binom{n}{2}}} \cdot \binom{n}{k} \cdot 2^{\binom{n}{2} - \binom{k}{2}} = \binom{n}{k} \cdot 2^{-\binom{k}{2}} = \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot 2^{-\binom{k}{2}} \leq$$

$$\frac{n^k \cdot 2^{-\binom{k}{2}}}{k!} < \frac{(2^{k/2})^k \cdot 2^{-\binom{k}{2}}}{k!} = \frac{2^{\frac{k^2}{2} - \frac{k(k-1)}{2}}}{k!} = \frac{2^{k/2}}{k!}$$

As k increases, the number $\frac{2^{k/2}}{k!}$ decreases. If $k \geq 3$, we have $\frac{2^{k/2}}{k!} < \frac{1}{2}$.

We will consider the case $k = 2$ later separately.

Similarly, if $k \geq 3$, then $\mathbf{P}[O_k \hookrightarrow G] < 1/2$.

We had $P(G) \equiv$ „$K_k \not\hookrightarrow G$ and $O_k \not\hookrightarrow G$". If $k \geq 3$, we get

$$\mathbf{P}[K_k \not\hookrightarrow G \text{ and } O_k \not\hookrightarrow G] = 1 - \mathbf{P}[K_k \hookrightarrow G \text{ or } O_k \hookrightarrow G] \geq$$
$$1 - \mathbf{P}[K_k \hookrightarrow G] - \mathbf{P}[O_k \hookrightarrow G] > 1 - 1/2 - 1/2 = 0 \ .$$

Thus, if $k \geq 3$, we have $r(k,k) \geq 2^{k/2}$.

If $k = 2$, then $r(k,k) = 2 = 2^{k/2}$. $\square$

Exact values of $r(k,l)$ are known only for a few pairs (k,l). A dynamic survey can be found at
`http://www.combinatorics.org/Surveys/ds1.ps`