Planar graphs

Graph is *planar* (*tasandliline*), if it can be drawn in the plane so that its edges do not intersect outside the vertices.

Example: K_4 is planar, Q_3 is planar, $K_{3,3}$ is not.

This definition is not formally strict, because "drawing" is not a mathematical term.

Next we will give one mathematical definition of drawing, but we will use the intuitive one in what follows anyway.

A *curve* (*kõver*) in the Euclidean space $\mathbb{R}^n$ is a function $\gamma : [a, b] \longrightarrow \mathbb{R}^n$, where $a, b \in \mathbb{R}$.

The curve γ is *continuous* (*pidev*), if for every $y \in \mathbb{R}$ we have $\lim_{x \to y} \gamma(x) = \gamma(y)$.

The *length* of the curve γ is

$$\sup\{\sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)) \mid k \in \mathbb{N}, a = t_0 < t_1 < \ldots < t_k = b\} \ .$$

Jordan curve is a non-self-intersecting continuos curve that has a length (note that a curve is not guaranteed to have it). Let J_n be the set of all Jordan curves in the space $\mathbb{R}^n$.

A *drawing* of the graph $G = (V, E)$ in the space $\mathbb{R}^n$ is a pair of mappings

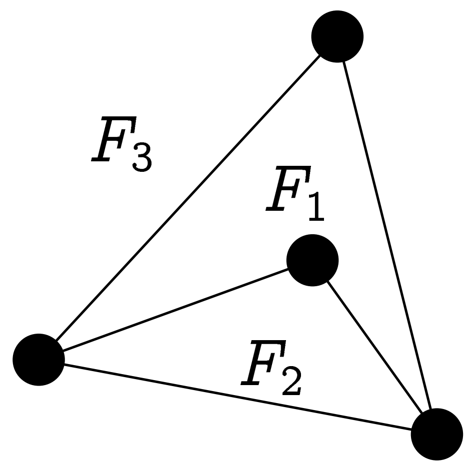
$$\iota_V : V \longrightarrow \mathbb{R}^n$$
$$\iota_E : E \longrightarrow J_n,$$

such that

- ι_V and ι_E are injective.
- If $\mathcal{E}(e) = \{u, v\}$, then the endpoints of $\iota_E(e)$ are $\iota_V(u)$ and $\iota_V(v)$.
- The curves $\iota_E(e_i)$ intersect each other only in their endpoints.

Graph is *planar*, if it has a drawing in the space $\mathbb{R}^2$.

The drawing of a graph partitions the portion of the plane not covered by the drawing.

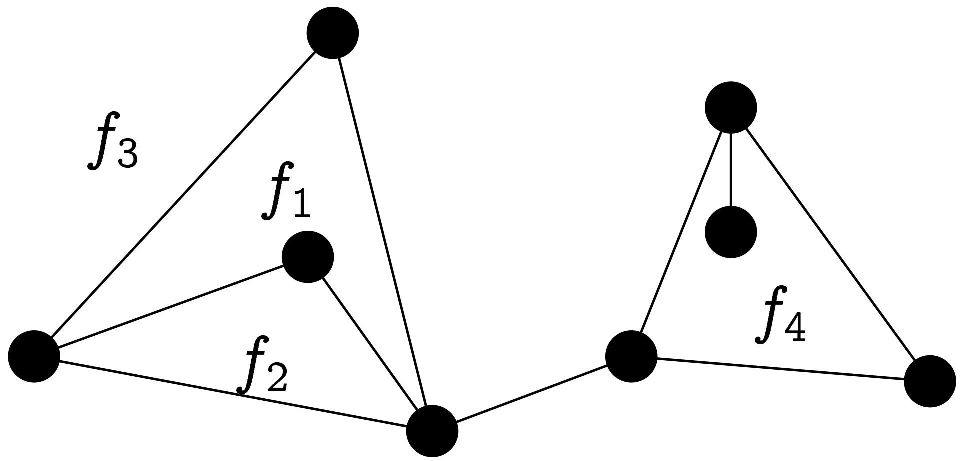


These parts are called *faces (tahud)*.

The face F_3 is *infinite face*.

A planar graph can be drawn in such a way that any face is infinite.

$\Rightarrow$ A planar graph be drawn so that any edge is outer.



Every face has a number of *sides (küljed)*.

- The number of sides: f_1 — 4, f_2 — 3, f_3 — 8, f_4 — 5.
- If some side has the same face on both sides, this side is counted twice.
- The number of all sides of all the faces is equal to the double of the number of edges in the graph.

Theorem (Euler). Let G be a connected planar graph.
Let

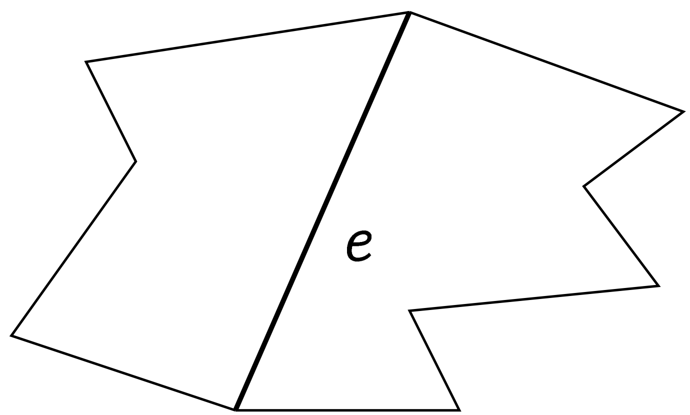
- n — the number of vertices of G,
- m — the number of edges of G,
- f — the number of faces of G.

Then $n + f - m = 2$.

Proof. Induction on the number of edges.

Base. G is a tree. Then $n = m + 1$ and $f = 1$. Thus
$n + f - m = m + 1 + 1 - m = 2$.

Step. Let G be a graph having m edges and not being a tree. There exists an edge e such that when it is deleted, G still remains connected.



The graph $G - e$ has one edge and one face less than the graph G. According to the induction hypothesis, we have $n + (f - 1) - (m - 1) = 2$. This implies $n + f - m = 2$.□

Corollary. Let G be a planar graph. Let

- n — the number of vertices of G,
- m — the number of edges of G,
- f — the number of faces of G,
- k — the number of connected components of G.

Then $n + f - m = k + 1$.

Proof. Apply the previous theorem to every connected component of G. The infinite face is counted only once.

□

Corollary. If G is a simple planar connected graph having at least three vertices, then $m \leq 3n - 6$ (again, m is the number of edges and n is the number of vertices).

Proof. Each face of the drawing of such a simple graph has at least 3 sides. Every side belongs to two sides, hence

$$2m = \sum_{F \text{ is a face}} \langle \# \text{ of } F\text{'s sides} \rangle \geq 3f \quad .$$

Euler's theorem gives

$$2 = n + f - m \leq n + \frac{2}{3}m - m = \frac{3n - m}{3}$$

or $3n - m \geq 6$. □

Corollary. K_5 is not planar.

Proof. In the graph K_5, we have $n = 5$ and $m = 10$. If K_5 would be planar, then the previous corollary would imply $m \leq 3n - 6$ or $10 \leq 9$. $\square$

Corollary. If G is a simple planar connected graph having at least three vertices, but no cycles of length 3, then $m \leq 2n - 4$.

Proof. Each face of the drawing of such a simple graph has at least 4 sides. Every side belongs to two sides, hence $2m \geq 4f$. Euler's theorem gives

$$2 = n + f - m \leq n + \frac{1}{2}m - m = \frac{2n - m}{2}$$

or $2n - m \geq 4$. □

Corollary. $K_{3,3}$ is not planar.

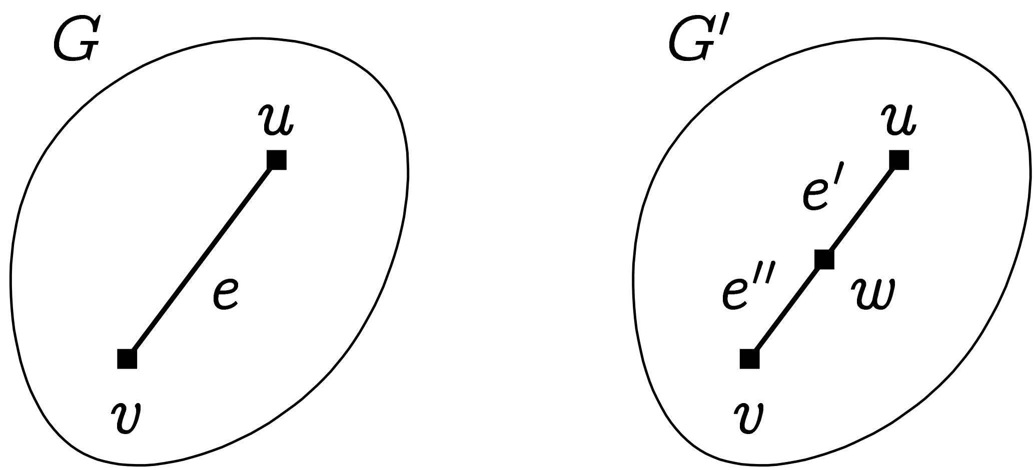
Tõestus. In the graph $K_{3,3}$ we have $n = 6$ and $m = 9$. Evenmore, $K_{3,3}$ has no cycles of length 3. If $K_{3,3}$ would be planar, then the previous corollary would imply $m \leq 2n - 4$ or $9 \leq 8$. $\square$

Corollary. Each planar simple graph has a vertex of degree at most 5.

Proof. Let G be a connected component of such a graph. Assume to the contrary that all the vertices of G have degree ≥ 6.

Since every edge is incident to two vertices, we have $6n \leq 2m$ or $m \geq 3n$. At the same time we have proven that $m \leq 3n - 6$. A contradiction. $\square$

The operation of *sudividing (poolitamine)* an edge ($G \Longrightarrow G'$):



The edge e is replaced by a the vertex w and edges e', e''.

Graphs G_1 and G_2 are *homeomorphic (homöomorfsed)*, if there exists a graph G such that G_1 and G_2 can be obtained from G by subdividing the edges.

Theorem (Kuratowski). A graph is planar iff it has no subgraphs homeomorphic to K_5 or $K_{3,3}$.

Stating it otherwise, graph G is not planar iff it “contains” K_5 or $K_{3,3}$ in the following way:

- The vertices of K_5 or $K_{3,3}$ are some vertices of G.
- The edges of K_5 or $K_{3,3}$ are some simple paths of G-s.
- These paths do not intersect anywhere but in the vertices.

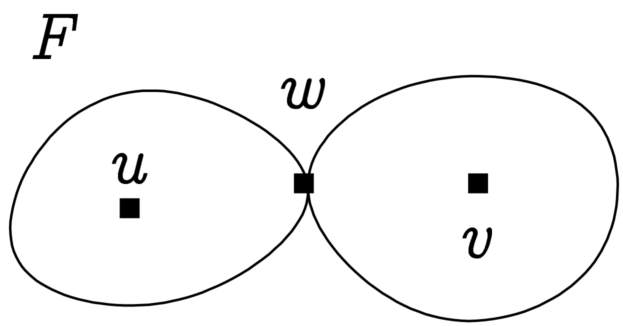
Proof. Assume to the contrary that there exist non-planar graphs that do not contain K_5 nor $K_{3,3}$. Let G be such a graph and let its edge set cardinality be the smallest possible.

The following holds true for G:

- G is a simple graph.
- G is connected.
- G has no bridges.
- G has no cut-vertices.

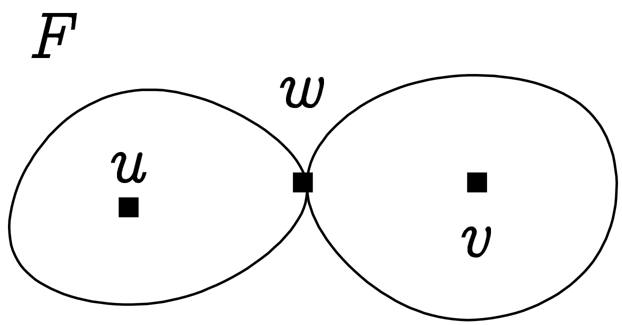
Let e be one of the edges of the graph G and let $\mathcal{E}(e) = \{u, v\}$. Let $F = G - \{e\}$. Then F is planar, since it does not contain K_5 nor $K_{3,3}$.

Claim 1. The graph F has no vertex w, such that F would be of the form

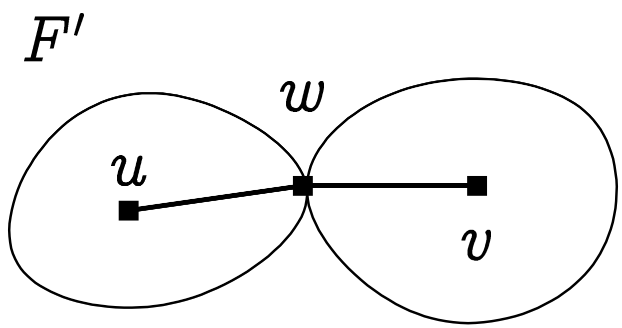


i.e. $F \backslash w$ would have u and v in different connected components.

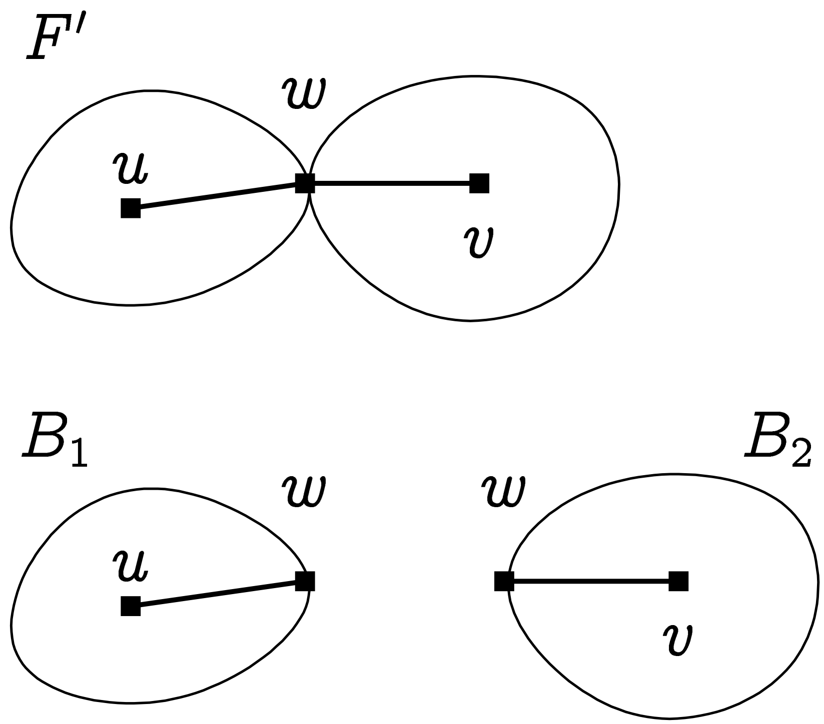
Assume to the contrary that F has the form



Let F' be the graph obtained from F by adding two edges:



Let B_1 and B_2 be the following graphs:



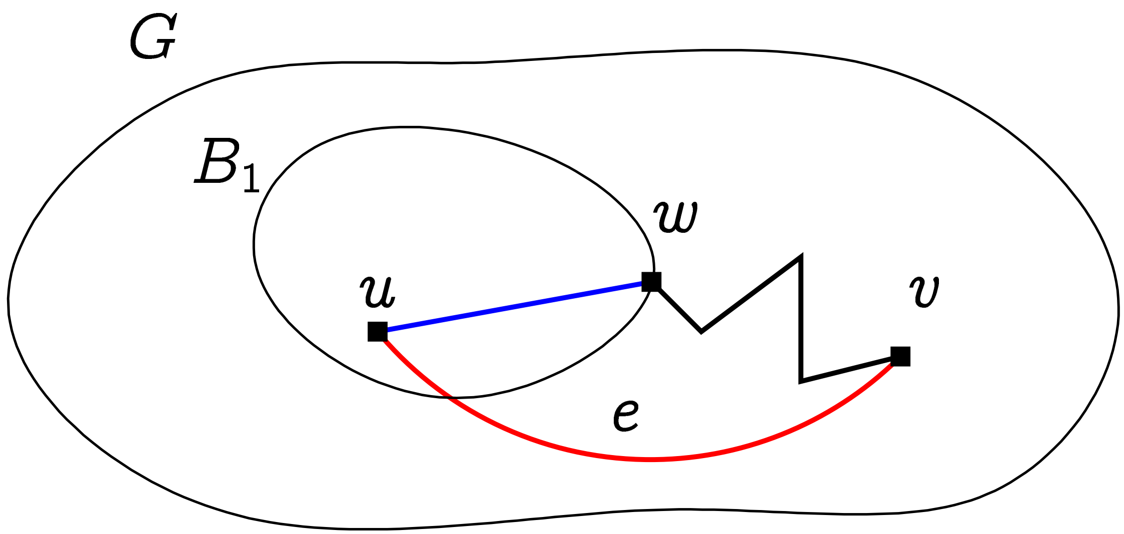
Graphs B_1 and B_2 have less edges than G, thus they satisfy the claim of the theorem

There are two options:

1st option. B_1 (or B_2) contains either K_5 or $K_{3,3}$.

This embedding must use the edge between u and w.

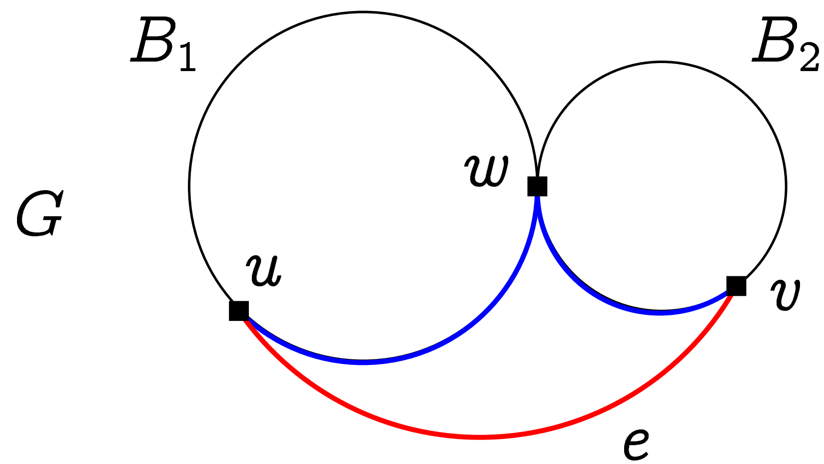
Then G contains either K_5 or $K_{3,3}$:



The new edge can be replaced by a simple path outside B_1.

2nd option. B_1 and B_2 are planar.

Then G is planar as well: draw B_1 and B_2 so that the new edges would be outer:



Claim 1 is proven.

Claim 2. There exists a cycle containing both u and v.

First make some observations about u (and v).

- u is not a cut-vertex of F (as $F\backslash u = G\backslash u$, it would then also be a cut-vertex of G)
- u and v are not neighbours in F (otherwise there would be a multiple edge between them in G)
- u has at least two neighbours in F (if it had only one, removing it would cause u and v ending up in different connected components)
- Consequently, the edges incident with u can not be bridges

Let $U \subseteq V(F)\backslash\{u\}$ be the set of all vertices that are on some cycle together with u.

Assume to the contrary that $v \notin U$.

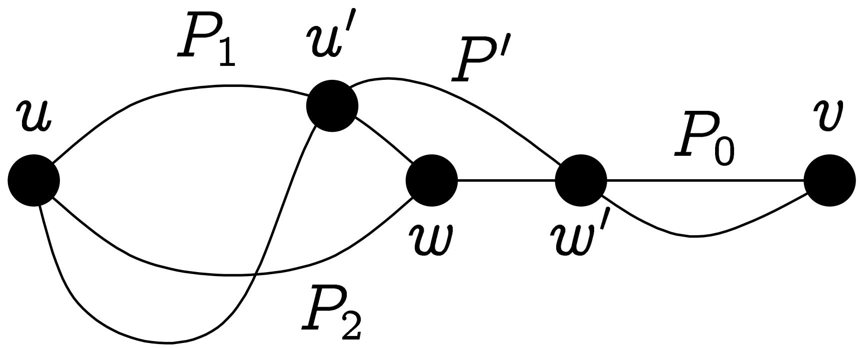
First we prove that $U \neq \emptyset$.

Let u' be a neighbour of u in F. Since $\{u, u'\}$ is not a bridge, there exists a path $u \rightsquigarrow u'$ in $F\backslash\{u, u'\}$. This path together with $\{u, u'\}$ is a cycle proving that $u' \in U$.

Let $w \in U$ be the vertex having the minimal distance from v. Let

- P_0 – the shortest simple path from w to v;
- P_1, P_2 – non-intersecting simple paths from u to w.

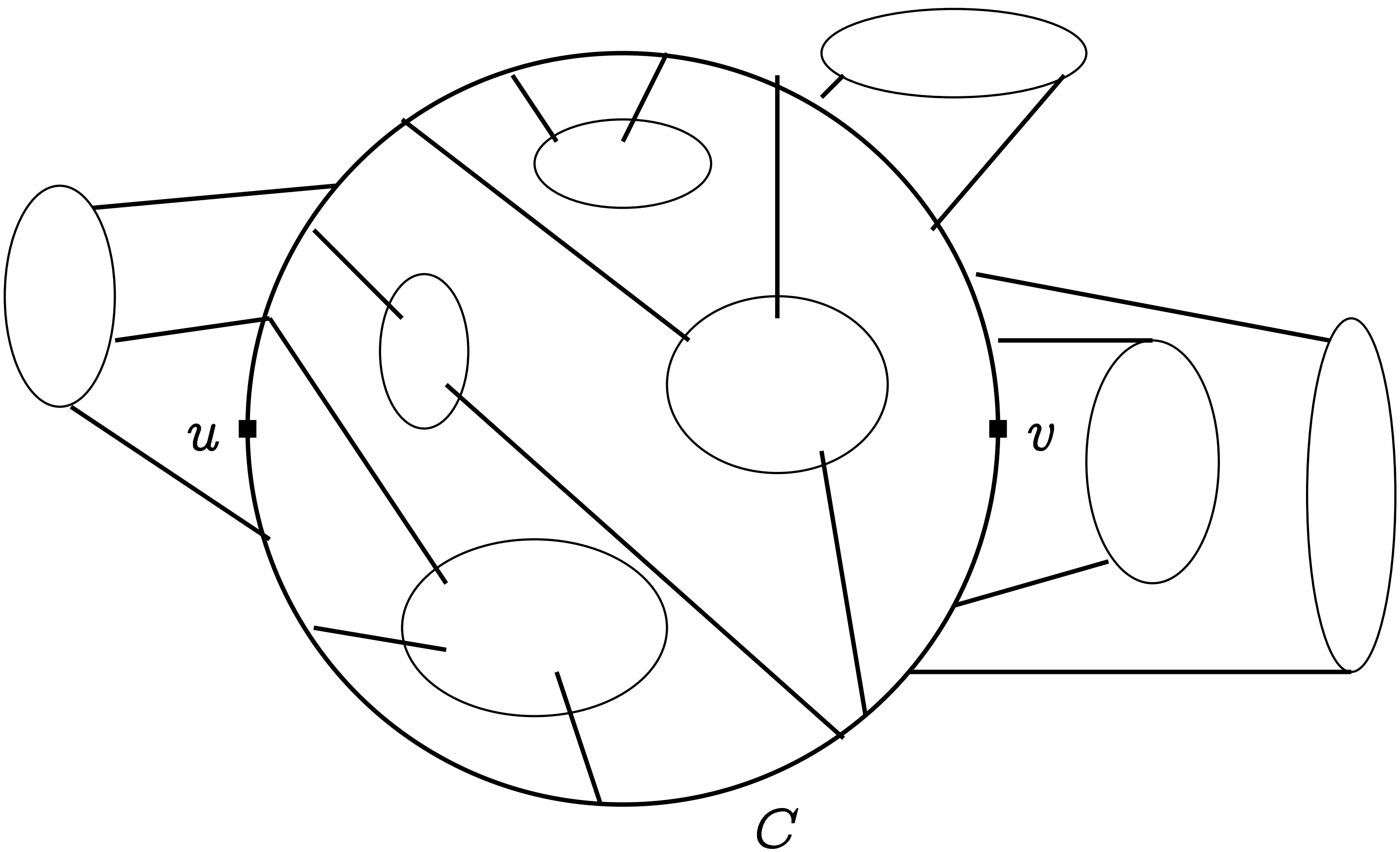
Due to the choice of w, P_0 does not intersect P_1 and P_2.



- Let P' be a simple path $u \rightsquigarrow v$ not passing through w (it exists due to Claim 1, since otherwise removing w would disconnect u and v);
- Let w' be the first (starting from u) vertex on the path P' that is also on the path P_0;
- Let u' be the last (starting from u) vertex on the path P' before w' that is also on the path P_1 or P_2. W.l.o.g. assume that it is on P_1.

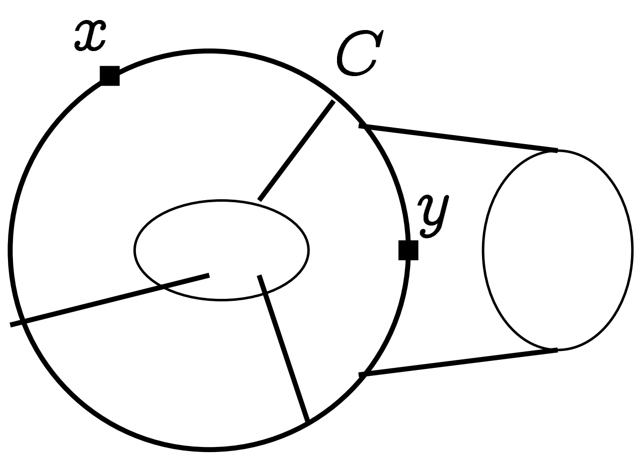
$u \overset{P_2}{\rightsquigarrow} w \overset{P_0}{\rightsquigarrow} w' \overset{P'}{\rightsquigarrow} u' \overset{P_1}{\rightsquigarrow} u$ is a cycle, thus $w' \in U$ and $d(w', v) < d(w, v)$. Contradiction with the choice of w.

Let F be drawn in the plane and let C be a cycle containing u and v. Choose the drawing and the cycle so that the number of faces remaining inside C is as large as possible.

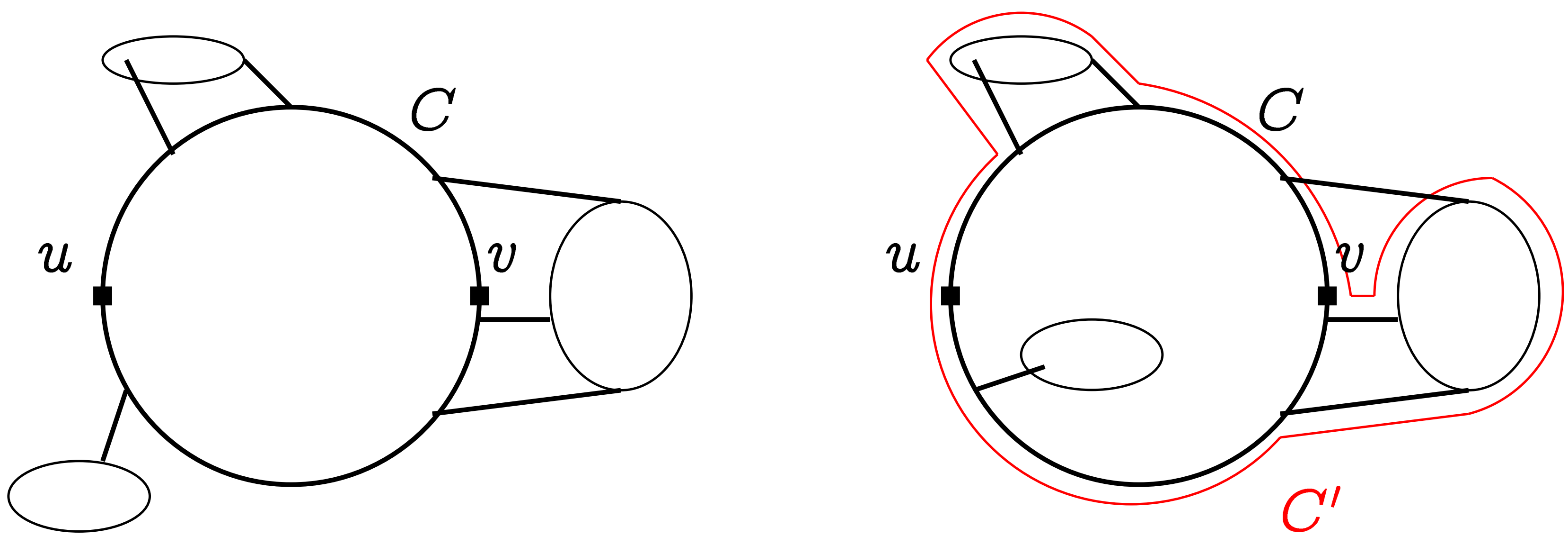


Besides the cycle C the graph F has more *components*. Some of them are *inner*, some *outer*.

Let x and y be two vertices on C. We say that some inner/outer component *separates* x and y if it is on the way when drawing a line from x to y inside/outside C.

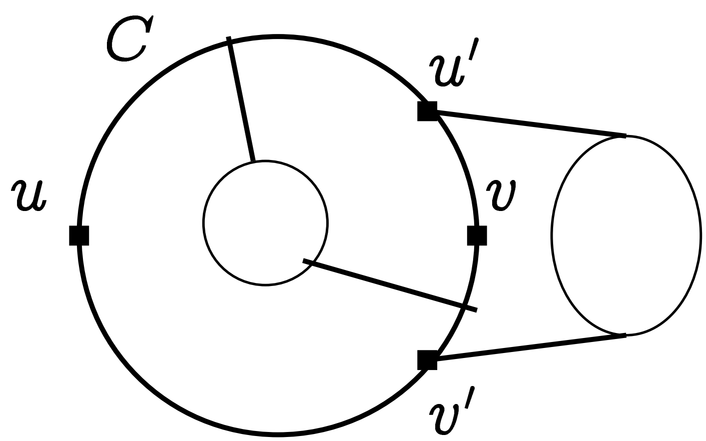


All the outer components seprate u and v and are joined with C by exactly two edges:

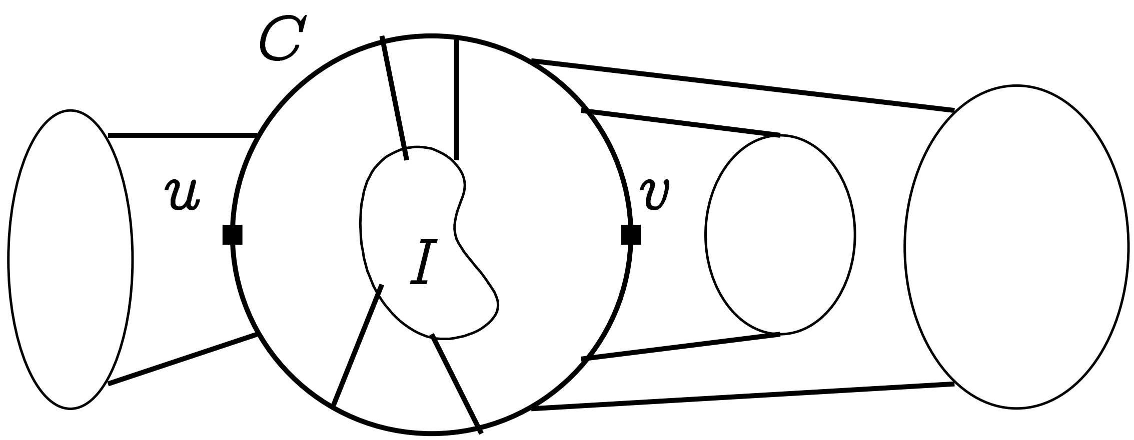


Otherwise we would get another drawing / cycle containing more faces.

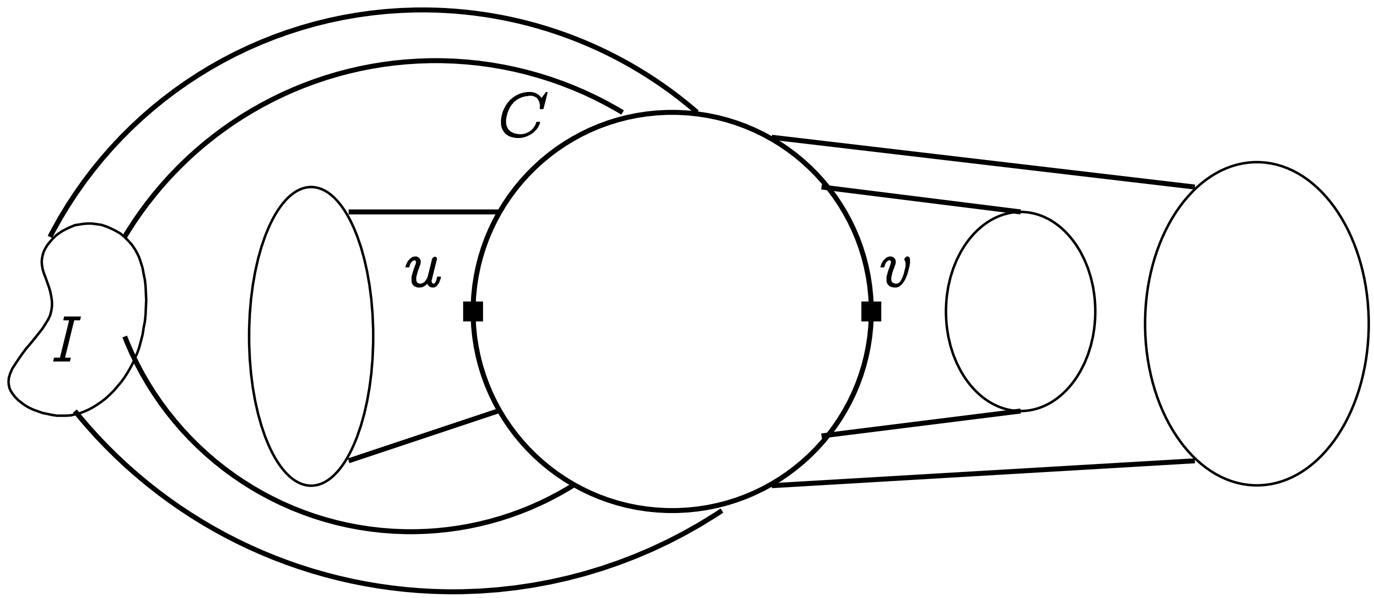
Claim 3. There exist an inner component and an outer component (being joined with C at vertices u' and v') so that the inner component separates both u, v and u', v':



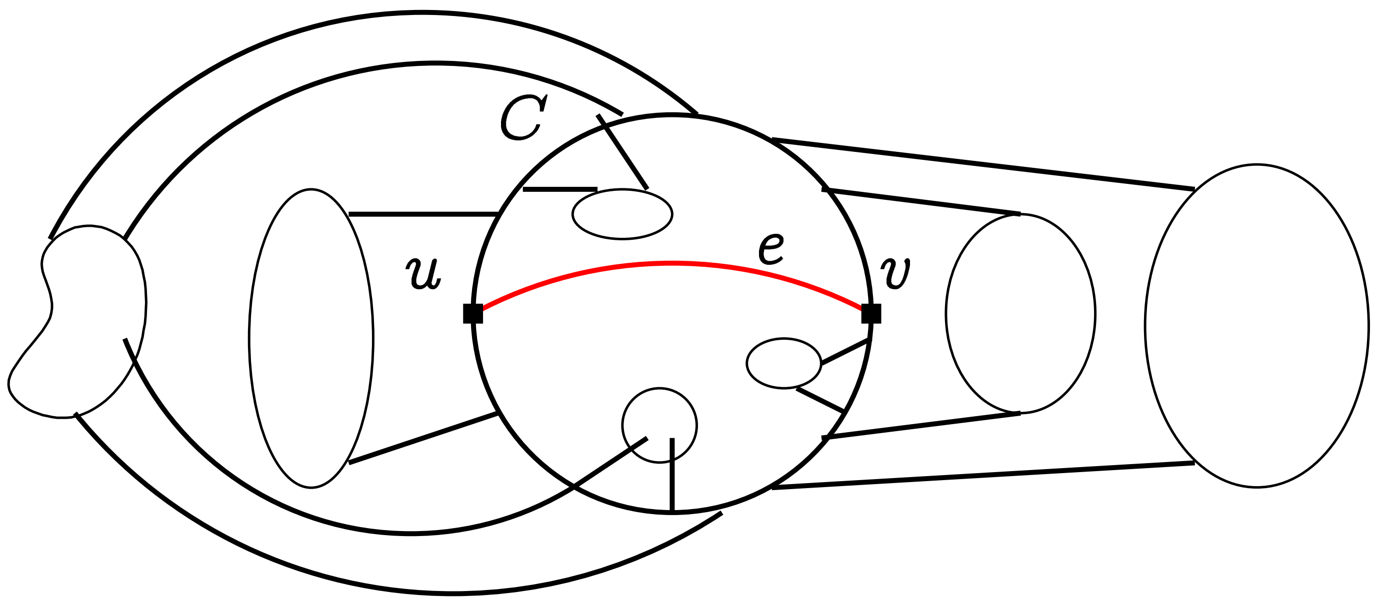
Proof of the claim: Let I be an inner component separating u and v such that it does not separate any two vertices where some outer component is joined with C:



We can take I out:

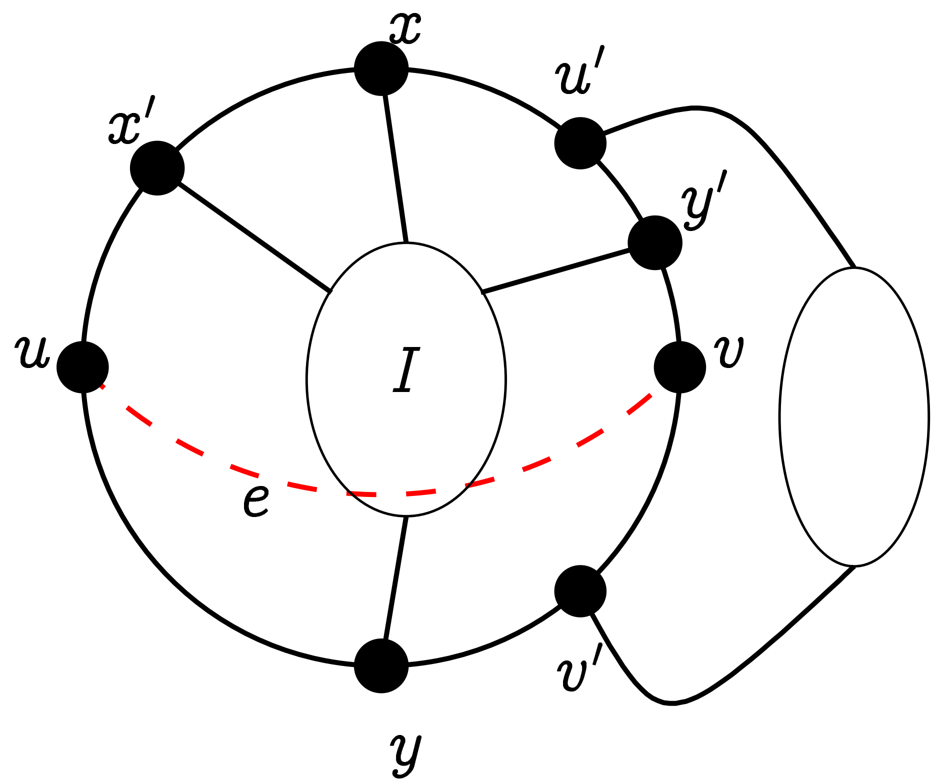


If the Claim 3 would not hold, we could take out all the inner components separating u and v. Then we can put back the edge e. Thus G would be planar; a contradiction. Thus the Claim 3 holds.



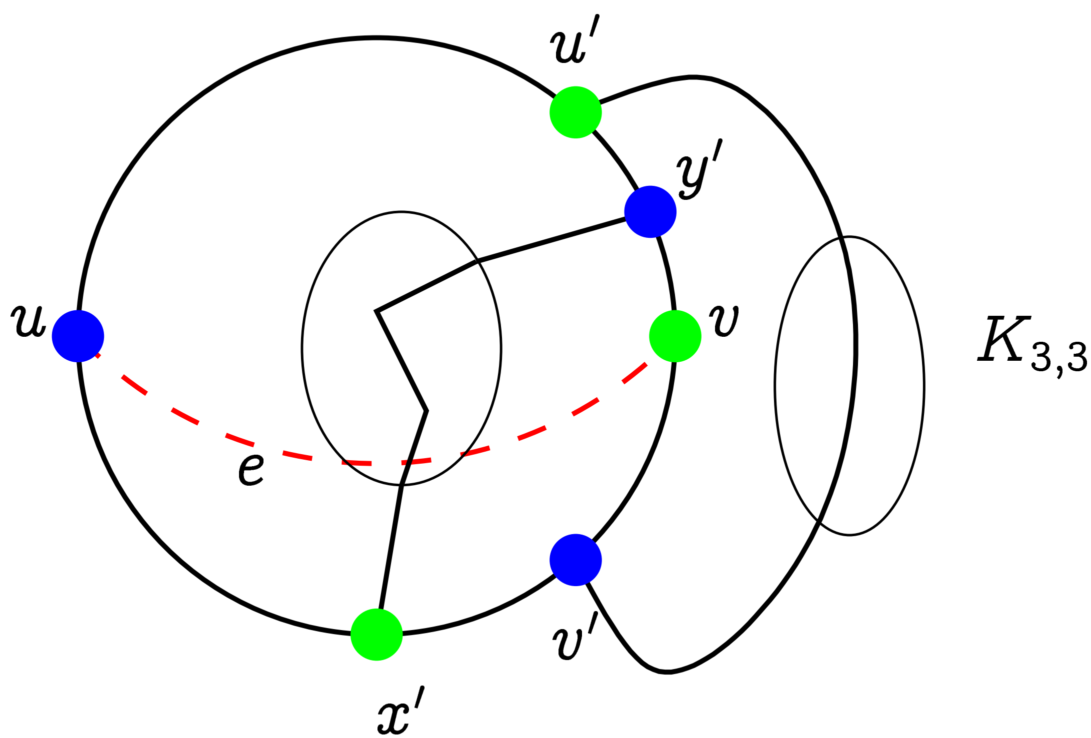
Let x, y be the vertices that I has separating u and v.

Let x', y' be the vertices that I has separating u' and v'.



They can be arranged in several ways. We will consider them and find K_5 or $K_{3,3}$ from G in all cases.

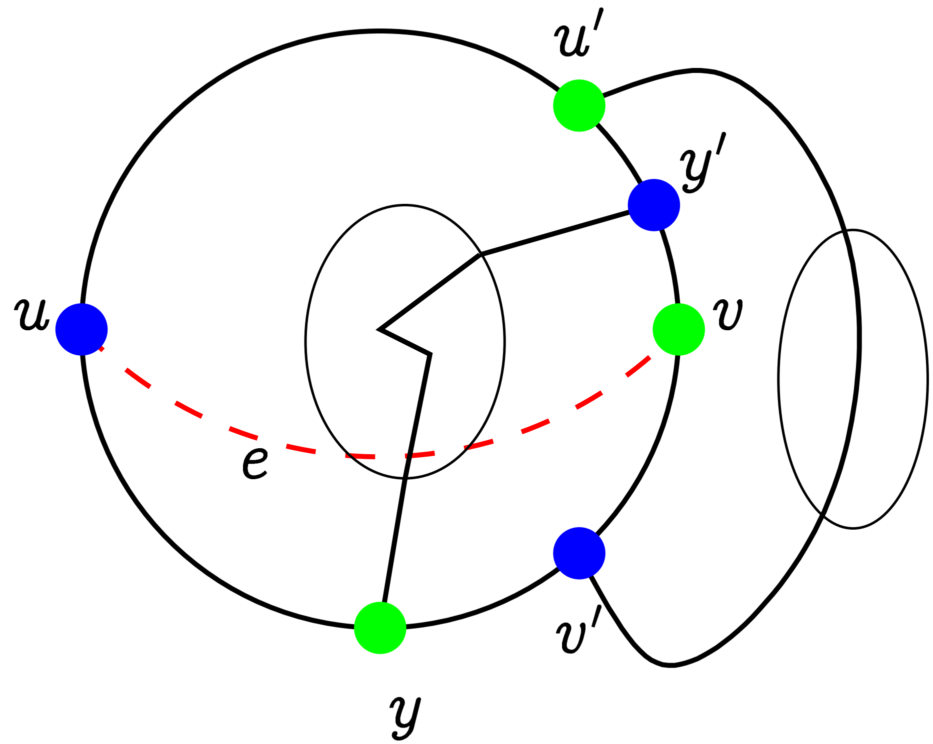
1st way. x', y' differ from u and v and I separates u and v due to x', y' as well.



2nd way x', y' differ from u and v and I does not separate u and v due to x', y'.

We can assume that x', y' are on the same side as x.

1st option. y is between u and v'.

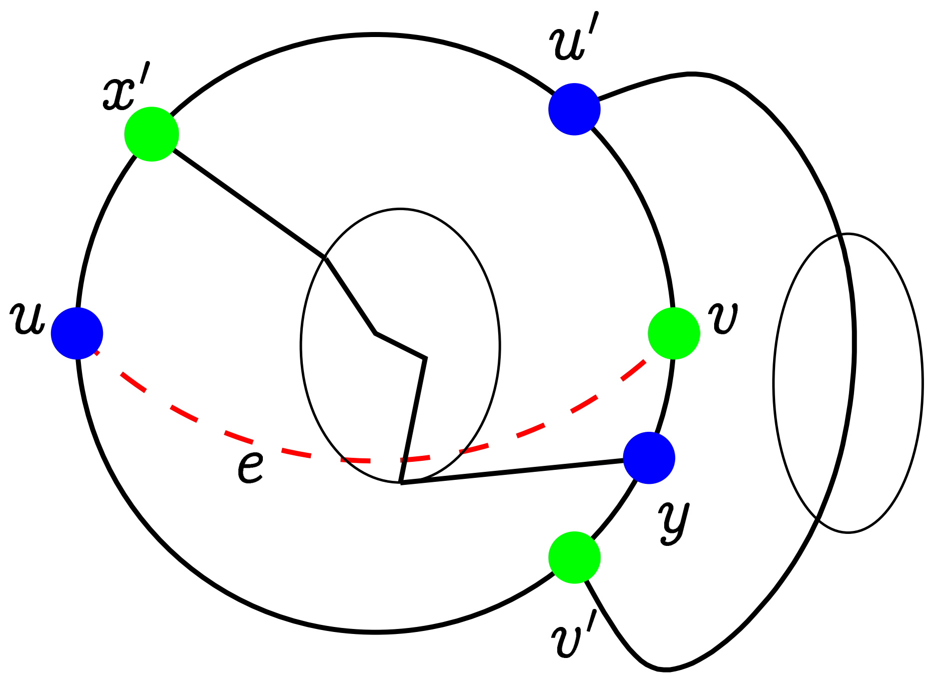


$K_{3,3}$

2nd way. x', y' differ from u and v and I does not separate u and v due to x', y'.

We can assume that x', y' are on the same side as x.

2nd option. y is between v' and v.

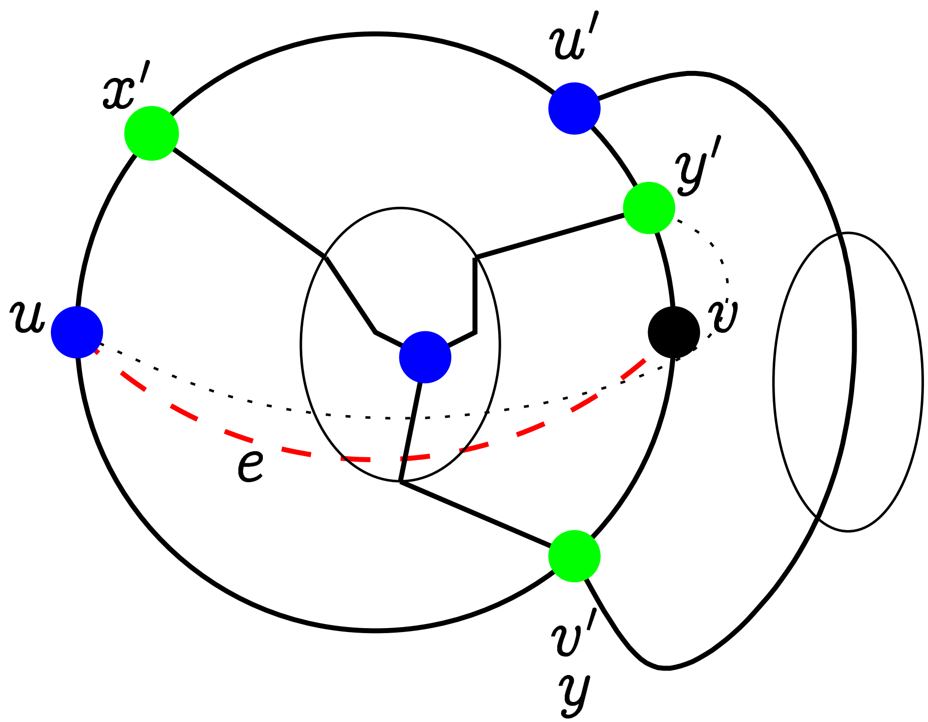


$K_{3,3}$

2nd way. x', y' differ from u and v and I does not separate u and v due to x', y'.

We can assume that x', y' are on the same side as x.

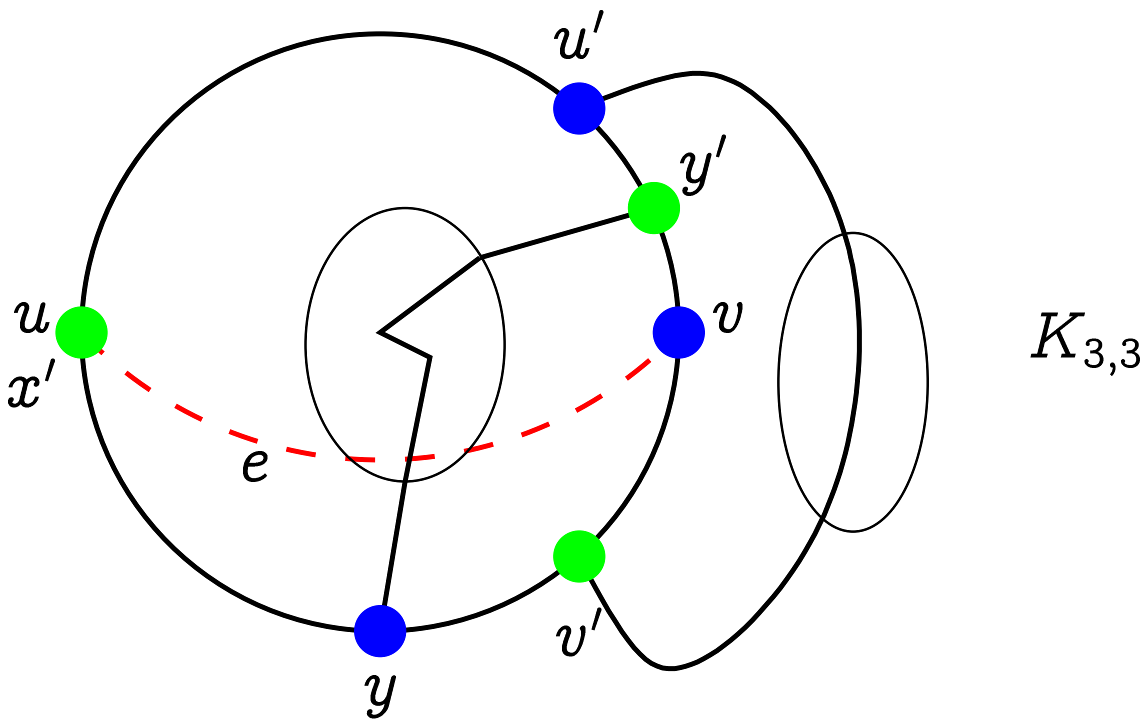
3rd option. $y = v'$.



$K_{3,3}$

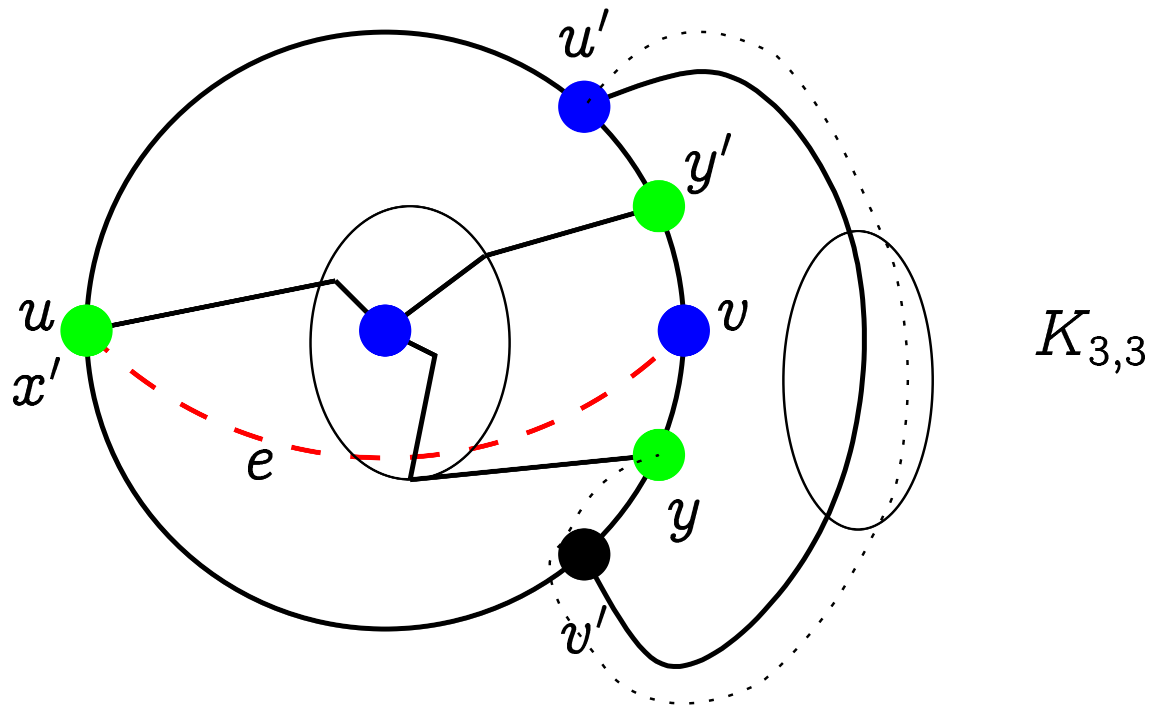
3rd way. $x' = u$ and $y' \neq v$. Assume that y' is between u' and v.

1st option. y is between u and v'.

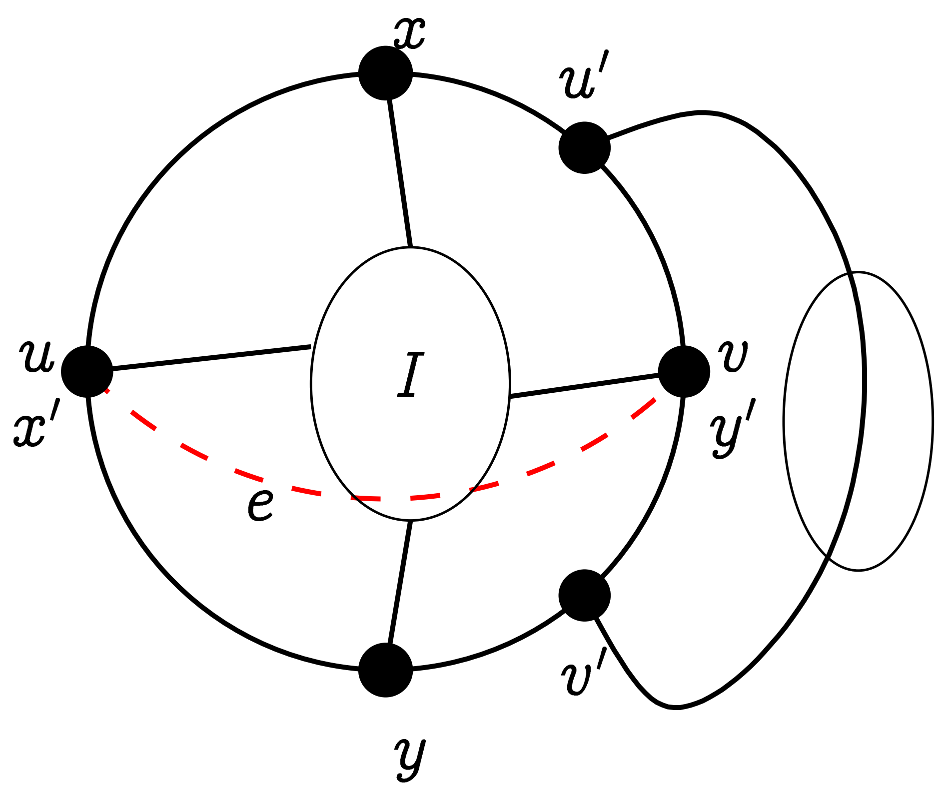


3rd way. $x' = u$ and $y' \neq v$. Assume that y' is between u' and v.

2nd option. y is between v' and v or $y = v'$.



4th way. $x' = u$ and $y' = v$.

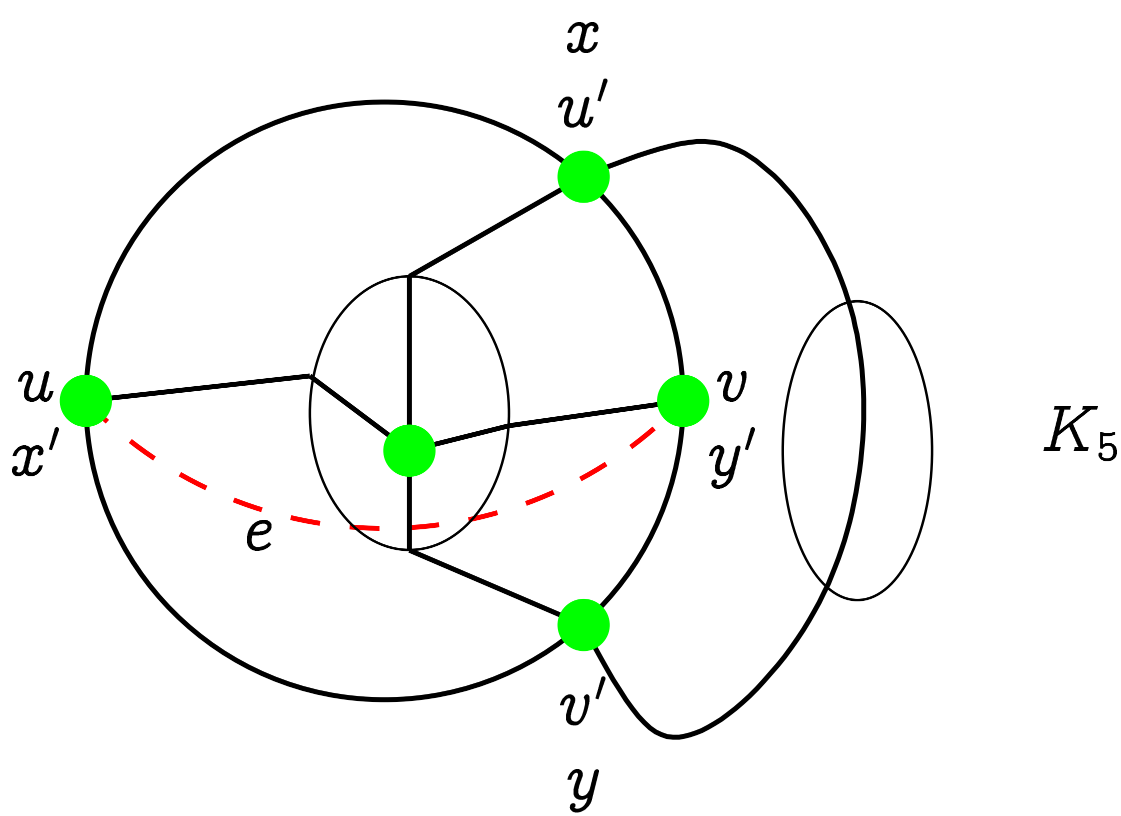


If x and y are not u' and v', then we exchange the notations ($u \leftrightarrow u'$, $v \leftrightarrow v'$, $x \leftrightarrow x'$, $y \leftrightarrow y'$, $e \leftrightarrow$ the path outside C). We are back to one of the three first ways.

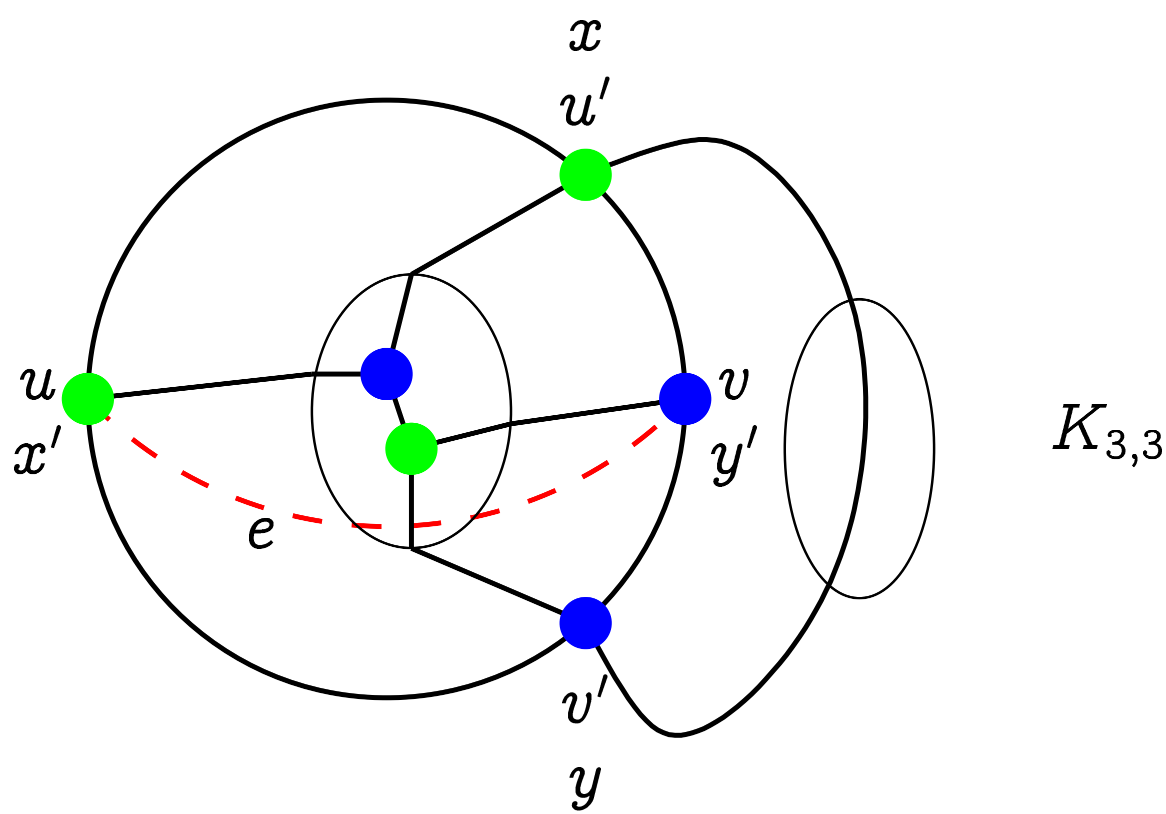
We are left with the case $x' = u$, $y' = v$, $x = u'$, $y = v'$.

The vertices neighbouring u, v, u', v' within the inner component are connected somehow within the component.

The first possible connection:

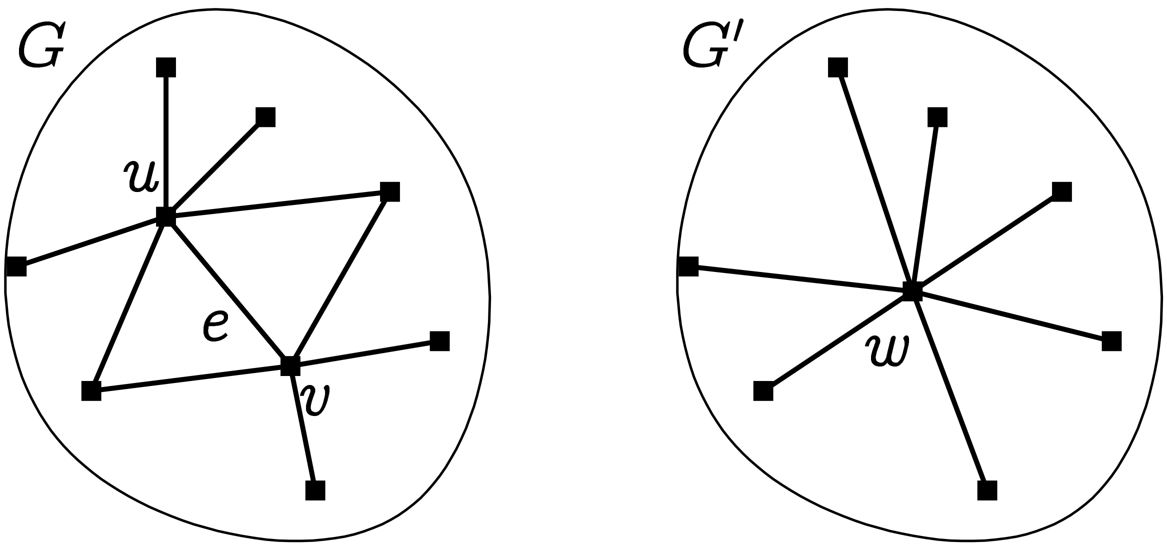


The second possible connection:



The theorem is proven. □

Edge *contraction (kokkutõmbamine)* ($G \Longrightarrow G'$):



When edges are contracted, a planar graph remains planar.

Theorem (Wagner). A graph is planar iff it has no subrgaphs contractible to K_5 or $K_{3,3}$.

Proof. If G is planar, then all its subrgaphs are planar. If we contract edges in a planar subgraph, we still get a planar graph, thus we can't get K_5 or $K_{3,3}$.

If G is not planar then there exists $H \leq G$ such that H is homeomorphic to K_5 or $K_{3,3}$. Contracting the edges we can reverse the effect of subdividivision. $\square$