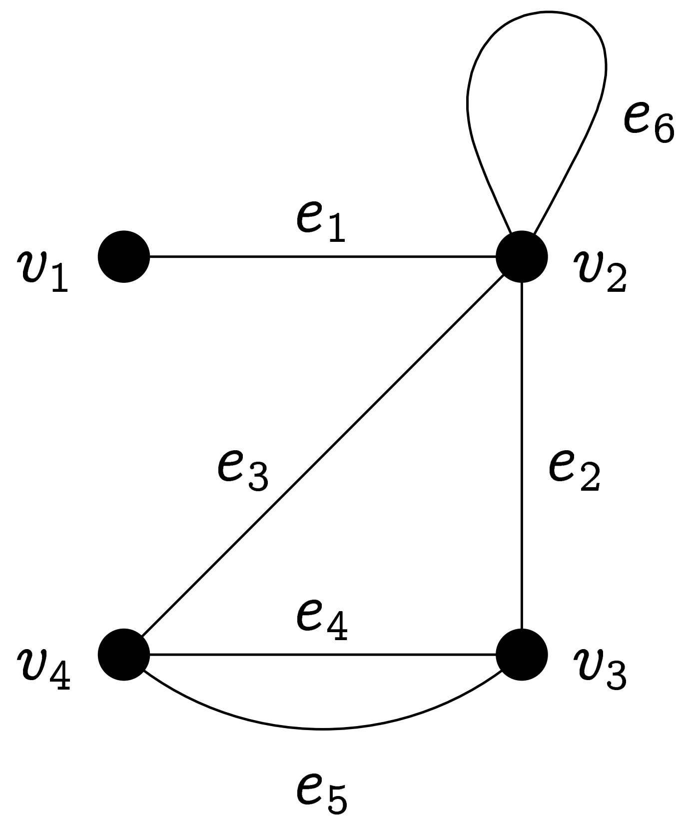
(Undirected) graph G can be defined as consisting of

- vertex set V,
- edge set E,
- incidence function $\mathcal{E} : E \longrightarrow \mathcal{P}(V)$, so that for all $e \in E$, the set $\mathcal{E}(e)$ of endpoints of e has either 1 or 2 elements.

In this course, we assume that V and E are finite and $V \neq \emptyset$.

Example: let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and

e	$\mathcal{E}(e)$
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_4\}$
e_4	$\{v_3, v_4\}$
e_5	$\{v_3, v_4\}$
e_6	$\{v_2\}$



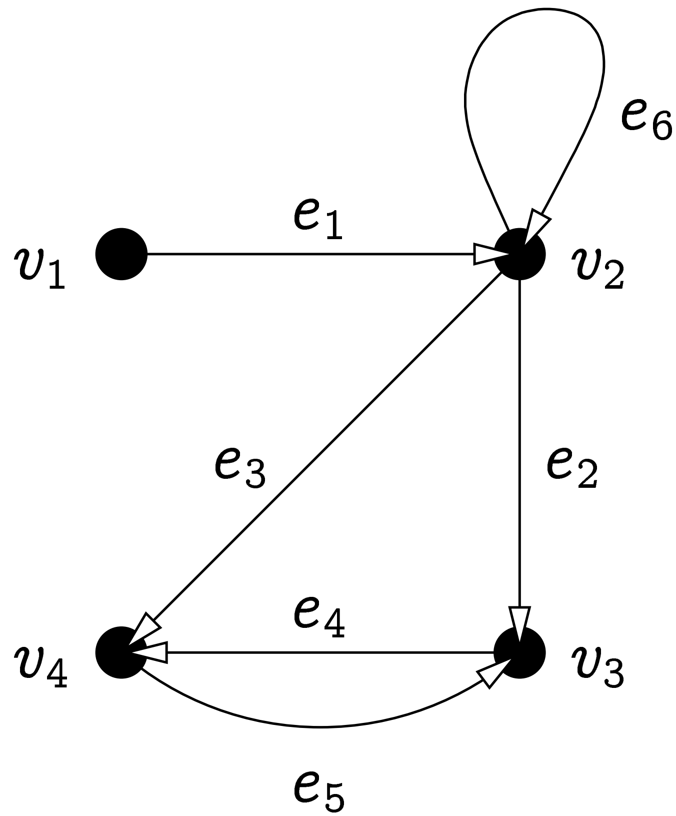
A graph may be illustrated using a figure.

Formally, graph is the triple $(V, E, \mathcal{E})$.

Directed graph consists of vertex set V, arc set E and incidence function $\mathcal{E} : E \longrightarrow V \times V$.

Example: let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and

e	$\mathcal{E}(e)$
e_1	(v_1, v_2)
e_2	(v_2, v_3)
e_3	(v_2, v_4)
e_4	(v_3, v_4)
e_5	(v_4, v_3)
e_6	(v_2, v_2)



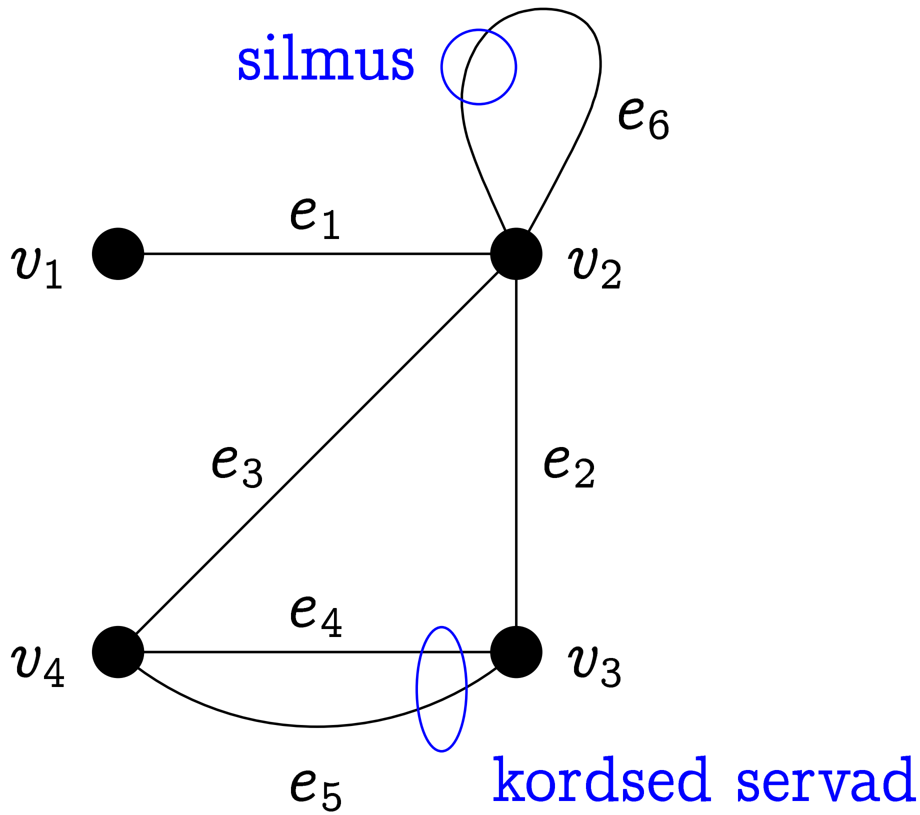
Let $G = (V, E, \mathcal{E})$ be a graph.

- If $v \in \mathcal{E}(e)$, we say that v and e are *incident*.
- It there exists e such that $\mathcal{E}(e) = \{v_1, v_2\}$ then v_1 and v_2 are *neighbours*.
- If $\mathcal{E}(e) = \{v_1, v_2\}$ then v_1 and v_2 are *endpoints* of the edge e. We also denote $v_1 \overset{e}{\text{---}} v_2$.

Let $G = (V, E, \mathcal{E})$ be a directed graph.

- If $\mathcal{E}(e) = (v_1, v_2)$ then v_1 and v_2 are called *initial vertex* and *terminal vertex* of the arc e, respectively.

$e \in E$ is a *multiple edge* if there exist $e' \in E \setminus \{e\}$ so that $\mathcal{E}(e) = \mathcal{E}(e')$. $e \in E$ is a *loop* if $|\mathcal{E}(e)| = 1$.



In a directed simple graph we can take $E \subseteq V \times V$.

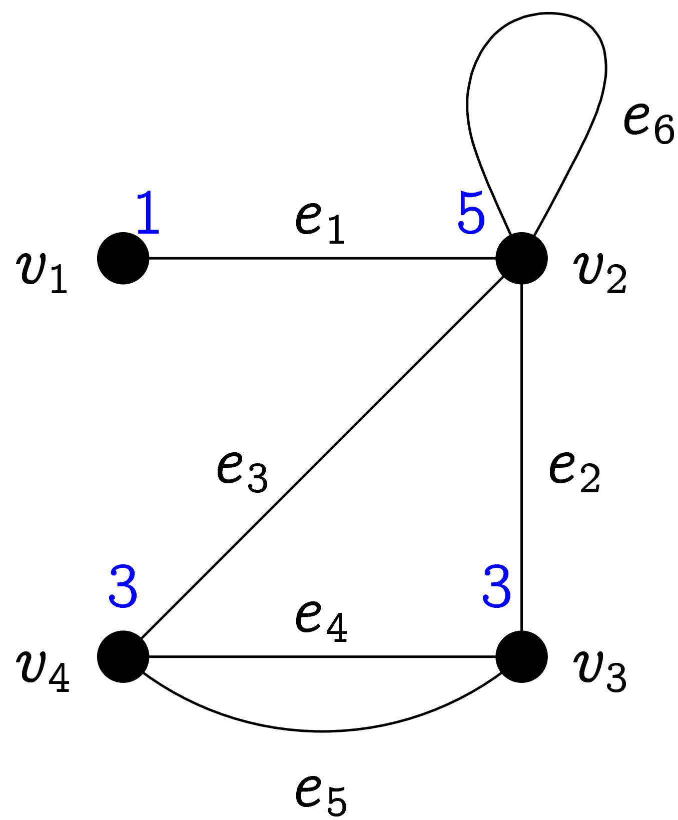
Example: let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$ and

e	$\mathcal{E}(e)$
e_1	(v_1, v_2)
e_2	(v_2, v_3)
e_3	(v_2, v_4)
e_4	(v_3, v_4)

Here we can take $E = \{(v_1, v_2), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$.

Degree deg(v) of the vertex $v \in V$ is the number of edges incident with it (where the loops are counted twice).

$$\deg(v) = |\{e \in E \mid v \in \mathcal{E}(e)\}| + |\{e \in E \mid \mathcal{E}(e) = \{v\}\}|$$



Theorem In a simple graph, there is an even number of verices having odd degree.

Proof. Let's count the total number of endpoints of all the edges in the simple graph $G = (V, E)$.

- On one hand, we get $2 \cdot |E|$.
- On the other hand, we get $\sum_{v \in V} \deg(v)$.

Since these quantities are equal, the sum of all vertex degrees is an even number. Thus we must have an even number of odd terms in the sum. $\square$

The same theorem holds if we allow loops and multiple edges.

In a directed graph $(V, E, \mathcal{E})$ we have two kinds of degrees for a vertex v:

- *indegree* $\overrightarrow{\deg}(v)$ — the number of arcs coming to the vertex v (i.e. the number of arcs having v as a terminal vertex); and
- *outdegree* $\overleftarrow{\deg}(v)$ — the number of arcs going from the vertex v (i.e. the number of arcs having v as a initial vertex).

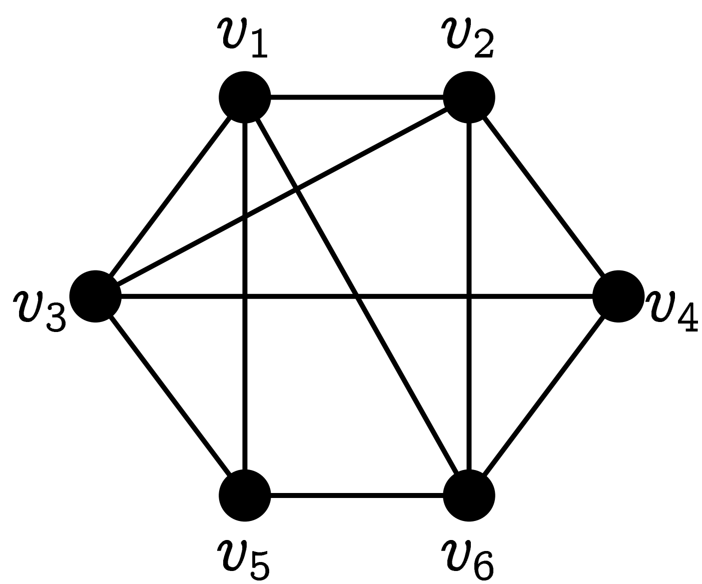
Again we can prove: $\sum_{v \in V} \overrightarrow{\deg}(v) = \sum_{v \in V} \overleftarrow{\deg}(v)$.

- A *walk* (from vertex x to vertex y) is the sequence

$$P : x = v_0 \overset{e_1}{\text{---}} v_1 \overset{e_2}{\text{---}} v_2 \overset{e_3}{\text{---}} v_3 \overset{e_4}{\text{---}} \ldots v_{k-1} \overset{e_k}{\text{---}} v_k = y \ .$$

- k is the length of the walk P, we also denote it as $|P|$.
- If P is a walk from x to y, we write $x \overset{P}{\rightsquigarrow} y$.
- A walk, having all the vertices different (except for possibly x_0 and x_k), is called a *path*.
- A walk with $v_0 = v_k$ is called a *closed walk*.
- A closed path is called a *cycle*.
- Graph is *connected* if there is a walk between any two of its vertices.
- *Distance* $d(u, v)$ between the vertices $u, v \in V$ is defined as the length of the shortest path between them.

Example



Walk: $v_1 — v_2 — v_4 — v_6 — v_2 — v_3$

Path: $v_1 — v_2 — v_3 — v_4$

Closed walk: $v_1 — v_2 — v_3 — v_1 — v_5 — v_6 — v_1$

Cycle: $v_1 — v_2 — v_6 — v_5 — v_1$

$d(v_1, v_4) = 2$, $d(v_1, v_2) = 1$, $d(v_1, v_1) = 0$.

Theorem If all the vertex degrees in a graph are at least 2, then there is a cycle in this graph.

Proof. Loop is a cycle. Multiple edges form a cycle.

Let $G = (V, E)$ be a simple graph. Take $v_1 \in V$. There exists $v_2 \in V$ such that $v_1 — v_2$. There exists $v_3 \in V$ such that $v_1 — v_2 — v_3$ is a path.

Let us have a path $v_1 — v_2 — \cdots — v_k$. There is a $v_{k+1} \in V$ so that $v_{k+1} \neq v_{k-1}$ and $v_k — v_{k+1}$.

If $v_{k+1} = v_i$ for some $i \in \{1, \ldots, k-2\}$ we have a cycle.

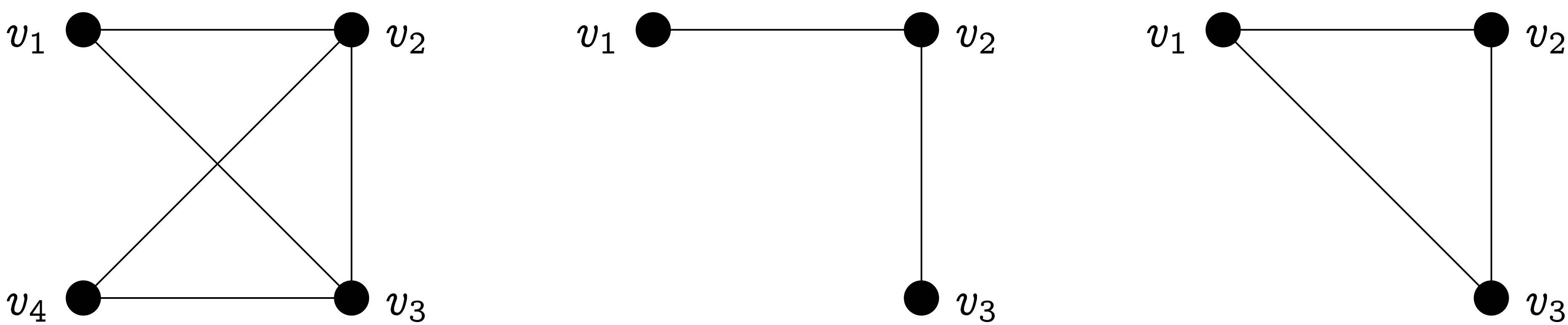
If not, we have a longer path. $v_1 — v_2 — \cdots — v_k — v_{k+1}$.

The length of this path is upper bounded by $|V|$ $\square$

$G' = (V', E')$ is a *subgraph* of graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$ and for every $e \in E'$ we have $\mathcal{E}(e) \subseteq V'$. We denote $G' \leqslant G$.

Subgraph (V', E') is said to be *induced* (by the set V'), if the set E' is the largest possible, i.e. for every $e \in E$ we have $\mathcal{E}(e) \subseteq V' \Rightarrow e \in E'$.

Example:

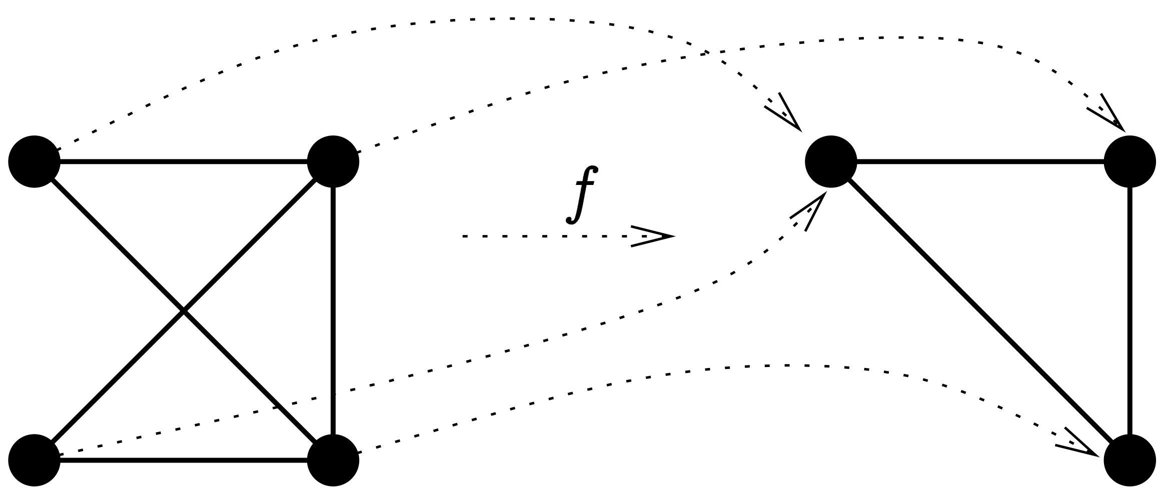


The maximal connected subraphs of a graph G are called its *connected compnents*.

More notions:

- An edge of the graph is called a *bridge*, if its removal increases the number of connected components.
- A vertex of the graph is called a *cut-vetrex*, if its removal increases the number of connected components.

Homomorphism from the graph $G_1 = (V_1, E_1)$ to the graph $G_2 = (V_2, E_2)$ is a mapping $f : V_1 \longrightarrow V_2$ such that the vertices $x, y \in V_1$ are neighbours iff the vertices $f(x), f(y) \in V_2$ are neighbours.



Homomorphism f is a *monomorphism* if it is one-to-one.

Homomorphism f is an *isomorphism*, if it is bijective.

Graphs G_1 and G_2 are *isomorphic* (denoted as $G_1 \cong G_2$) if there is an isomorphism between them.

- *Null graph* is a graph without edges. Null graph having n vertices is denoted as O_n or N_n.

- *Complete graph* is a graph having exactly one edge between each pair of vertices. Complete graph having n vertices is denoted as K_n.

Proposition. The graph K_n has $\frac{n(n-1)}{2}$ edges.

- The graph $G = (V, E)$ is called *bipartite* if V can be partitioned into two subsets V_1 and V_2 (i.e. $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$) so that no edge has its endpoints in the same subset V_i.

• A bipartite graph with bipartition V_1 and V_2 is *complete bipartite* if between each $v_1 \in V_1$ and $v_2 \in V_2$ there is an edge. If $|V_1| = m$ and $|V_2| = n$, we denote this graph $K_{m,n}$.

Proposition. Graph $K_{m,n}$ has mn edges.

Theorem A graph is bipartite $\Leftrightarrow$ all its cycles have even length.

Proof $\Rightarrow$. On a cycle, we must have vertices from the sets V_1 and V_2 alternating.

Proof $\Leftarrow$. Consider one connected component of $G = (V, E)$ (other components are handled similarly).

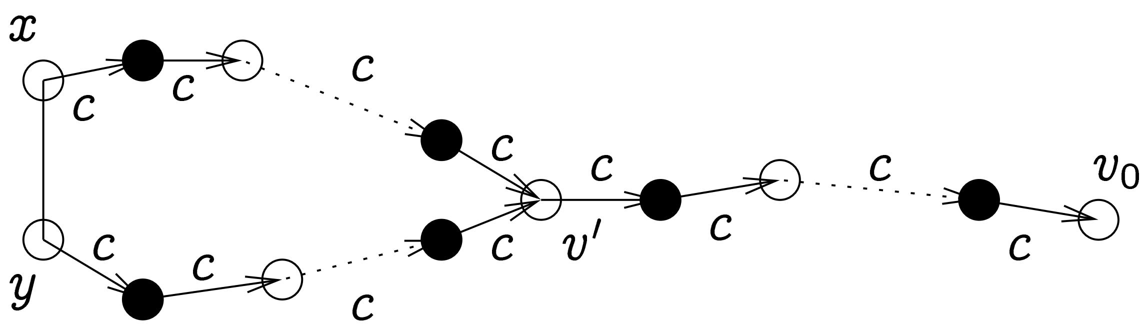
Colour the vertices of the graph G black and white.

Select a vertex $v_0 \in V$ and color it white.

Let u be a coloured vertex having uncoloured neighbours and let v be one of those. Colour v in a colour opposite to u's. Denote $v \stackrel{c}{\to} u$.

Repeat this procedure until we get two neighbours x and y coloured the same or until we run out of vertices.

If we have "bad" neighbours x and y, then



we have an odd cycle $x — \cdots — v' — \cdots — y — x$.

If we run out of the vertices, we have constructed a bipartition for this component. $\Box$