Eulerian graphs

Graph G is a pair (V, E), where V is the set of *vertices* and E is the set of *edges*. Besides that, we are given the *incidence function* $\mathcal{E}$.

Walk in the graph G is a sequence

$$v_0 \overset{e_1}{\text{---}} v_1 \overset{e_2}{\text{---}} v_2 \overset{e_3}{\text{---}} v_3 \overset{e_4}{\text{---}} \ldots v_{k-1} \overset{e_k}{\text{---}} v_k,$$

where $v_0, \ldots, v_k \in V$, $e_1, \ldots, e_k \in E$ and $\mathcal{E}(e_i) = \{v_{i-1}, v_i\}$.

The walk is *closed*, if its first and last vertices coincide.

Path is a walk where every vertex occurs at most once.

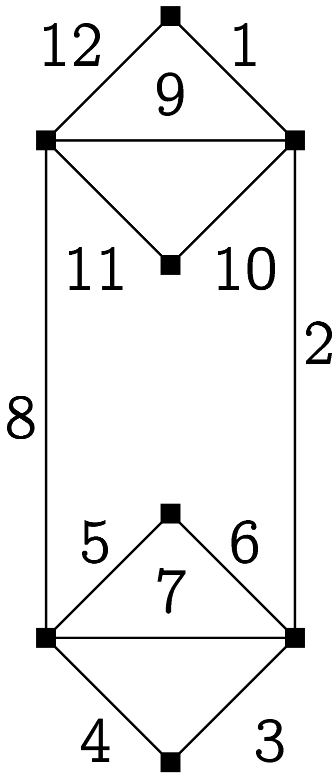
Cycle is a closed path.

Eulerian walk in the graph $G = (V, E)$ is a closed walk covering each edge exactly once.

Eulerian graph is a graph with a Eulerian walk.

A graph that has a non-closed walk covering each edge exactly once is called *semi-Eulerian*.

A well-known class of puzzles: draw the figure without raising the pen from the paper and covering each line exactly once.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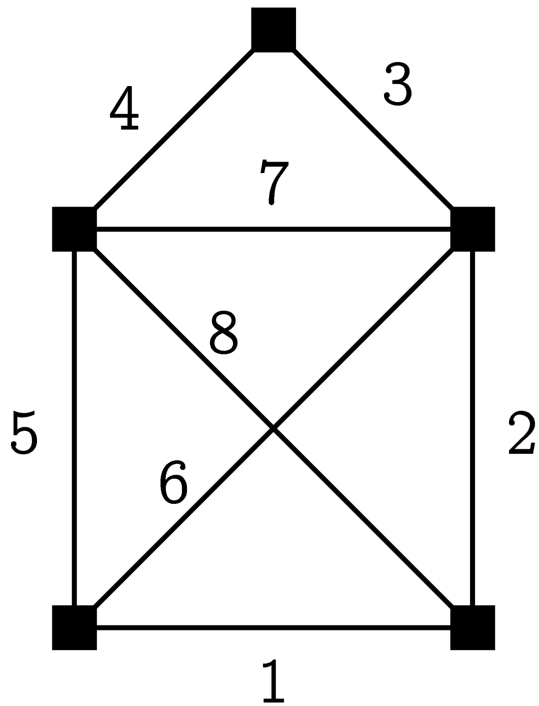
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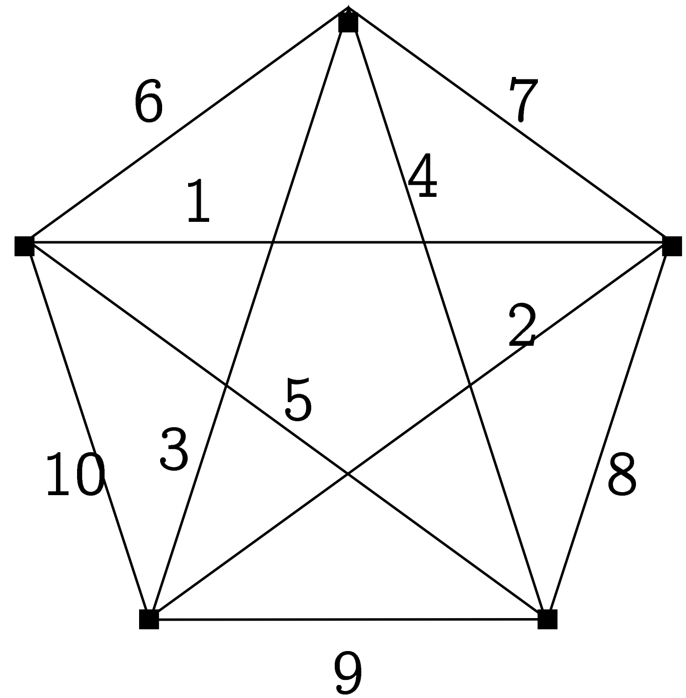
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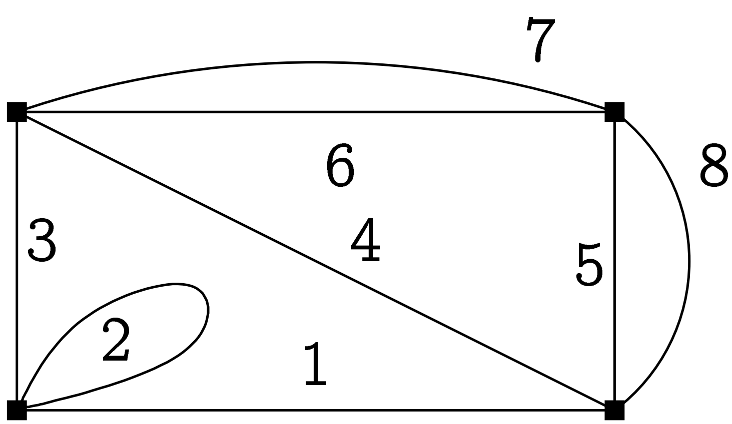
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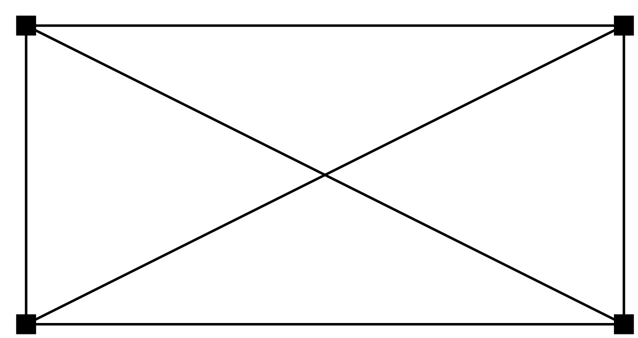
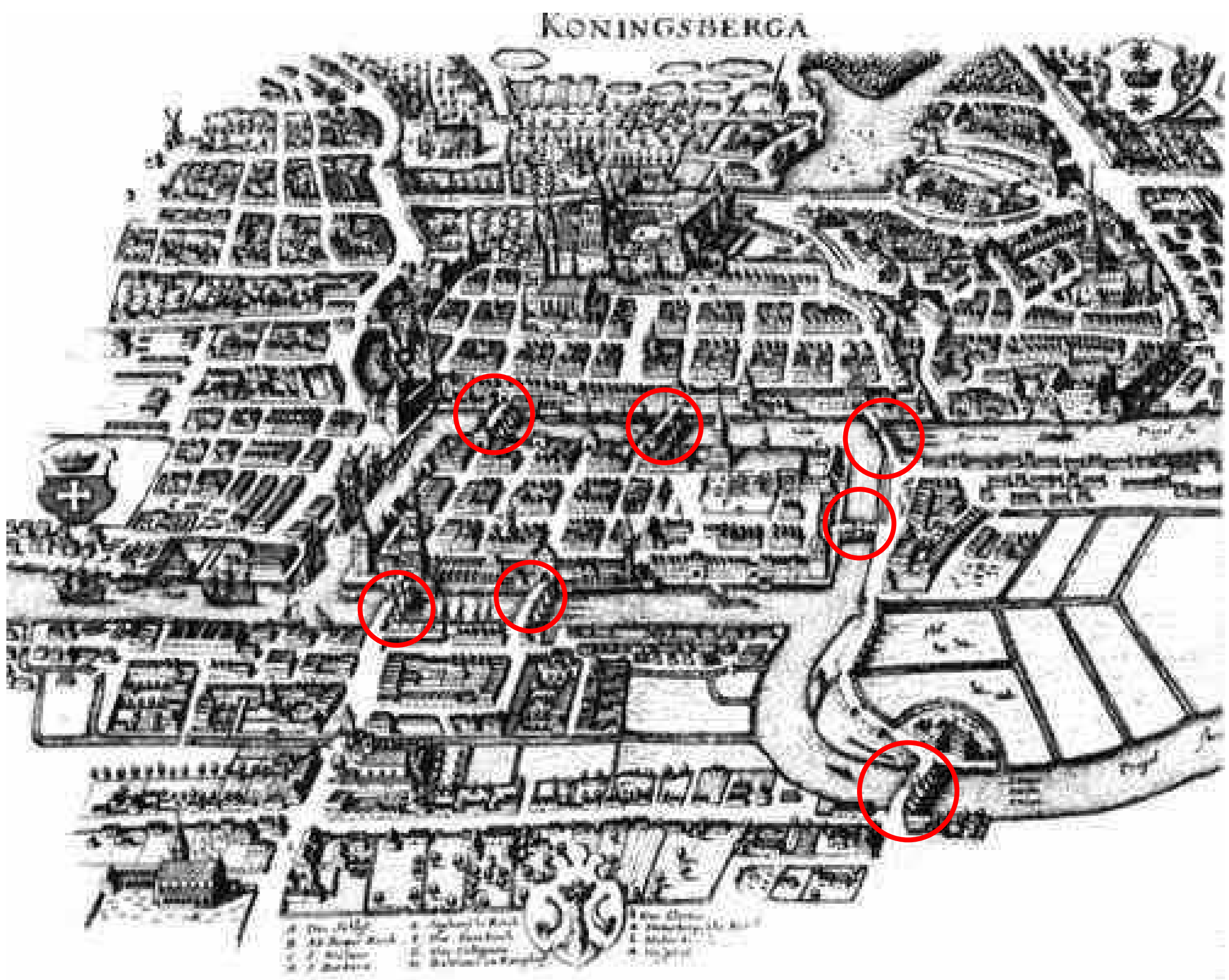
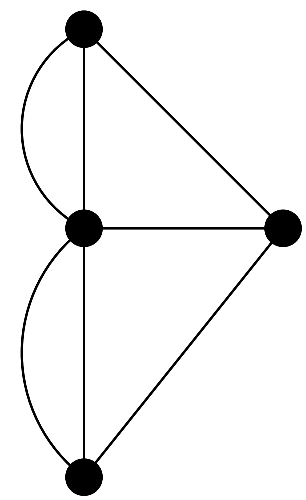


“Original problem”:





Theorem. Let $G = (V, E)$ be a connected graph. The following are equivalent:

(i). G is a Eulerian graph.

(ii). All vertex degrees of G are even.

(iii). E can be represented as a union of edge-wise non-intersecting cycles.

Proof (i)$\Rightarrow$(ii). Let P be some Eulerian walk of G and let $v \in V$.

The walk P enters v some number of times and also exits it the same number of times. Thus the number of edges of P incident with v is even (again, loops are counted twice).

On the other hand, P is a Eulerian walk, thus the edges of P incident with v are exactly all the edges of G incident with v.

Proof (ii)$\Rightarrow$(iii). Induction over $|E|$.

Base. $|E| = 0$. Then E is a union of 0 pieces, each one of them is ....

Step. $|E| > 0$. Since G is connected, all the vertex degrees must be positive.

According to (ii), all the vertex degrees are ≥ 2.

Using a theorem from the previous lecture, there is a cycle C in G.

Theorem. If all the vertex degrees in a graph are at least 2, then there is a cycle in this graph.

Delete all the edges of C from grapg G; let the remaining graph be G'.

G' has less edges than G and all its vertex degrees are still even.

Let $H_1, \ldots, H_k$ be the connected components of graph G'. Induction hypothesis implies that each of them can be represented as a union of edge-wise non-intersecting cycles. Adding the cycle C to the union of these representations, we have obtained the required representation for E.

Proof (iii)$\Rightarrow$(i). Let $E = C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup} C_n$, where $C_1, \ldots, C_n$ are cycles.

If $n = 1$, the claim is clear. Assume $n \geq 2$.

W.l.o.g assume that every cycle C_i $(i > 1)$ has a common vertex with some cycle C_j $(j < i)$.

We will now construct closed walks $P_1, \ldots, P_n$ so that each P_i covers each edge of the cycles $C_1, \ldots, C_i$ exactly once and does not cover any other edges.

Let the closed walk P_1 be the cycle C_1.

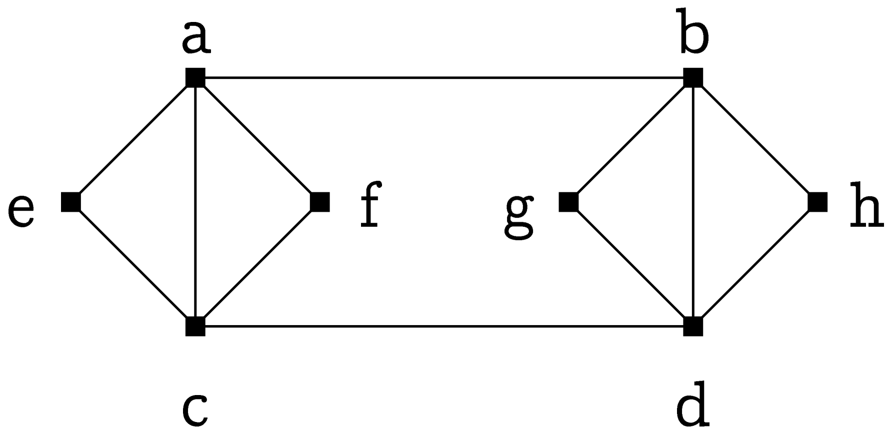
Construct the walk P_i based on the walk P_{i-1} as follows.

- Move along the walk P_{i-1} until we hit a vertex also present in the cycle C_i.
- Follow the cycle C_i starting and finishing in vertex v.
- Move along the rest of the walk P_{i-1}.

The walk P_n is a Eulerian one in graph G. $\square$

The proof gives an algorithm for finding a Eulerian cycle in a Eulerian graph G:

- Partition $E(G)$ into cycles.
  - Construct one of these cycles, say, C.
    - Move along the edges of G until we reach some vertex for the second time.
  - Remove the edges of C from graph G.
  - Partition the edges of the connected components of G (without C) to cycles.
  - Output these cycles and the cycle C.
- Construct a Eulerian walk as shown in the previous slide.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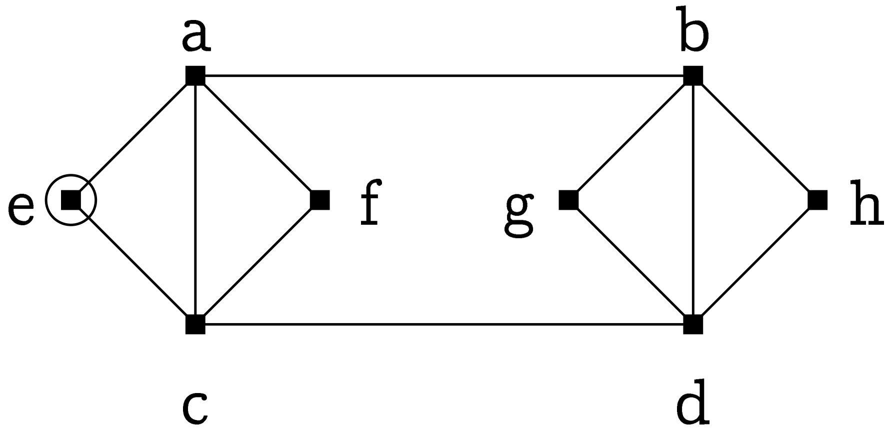
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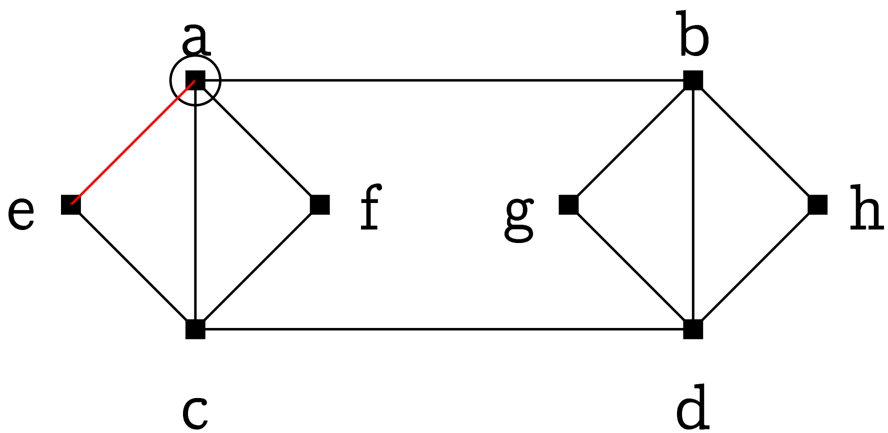
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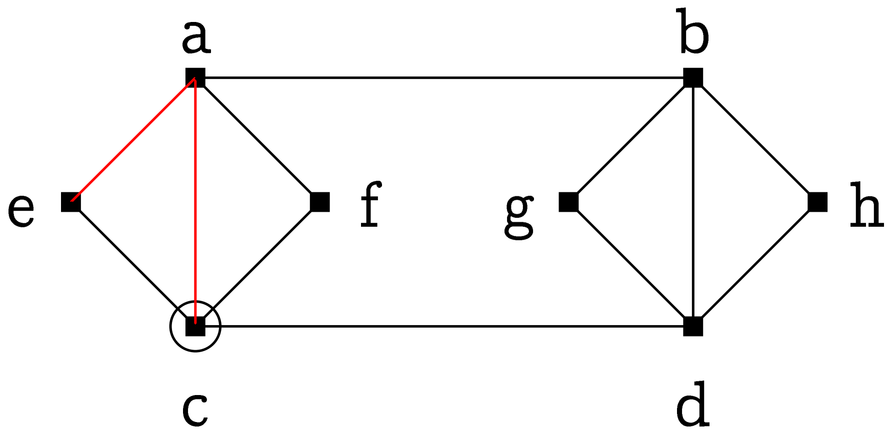
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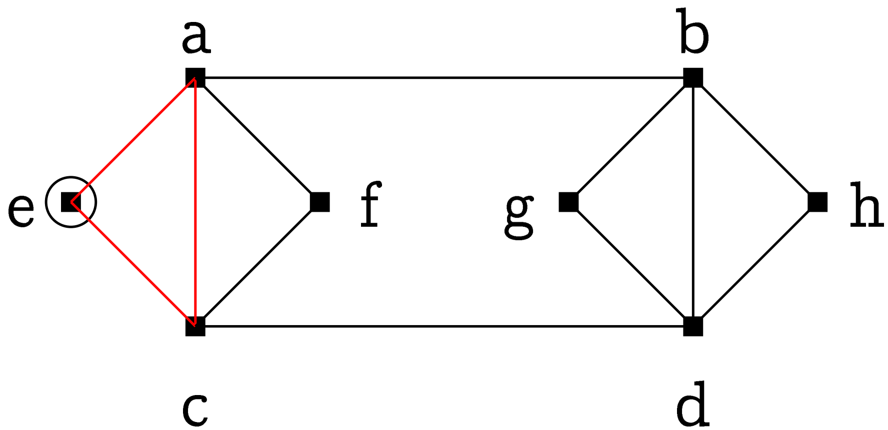
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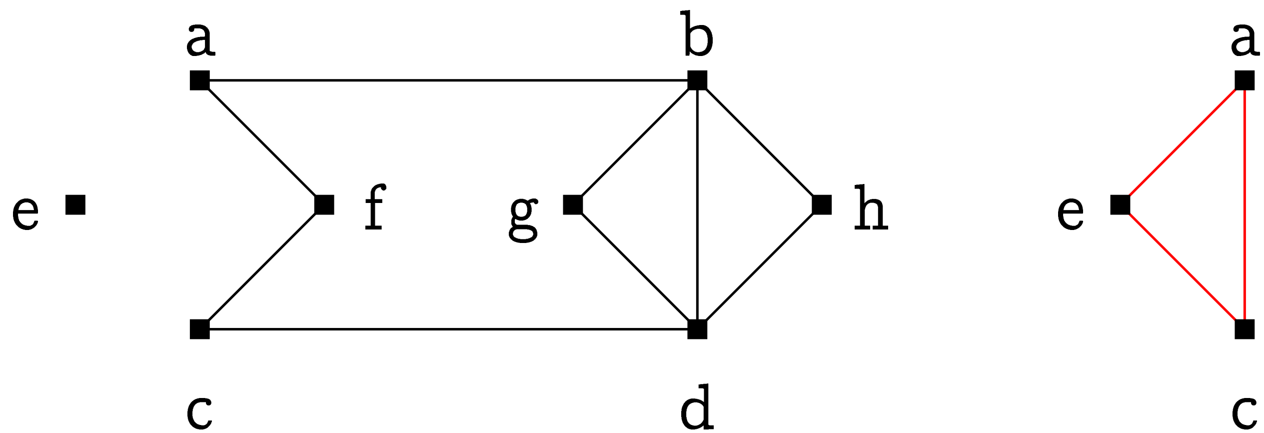
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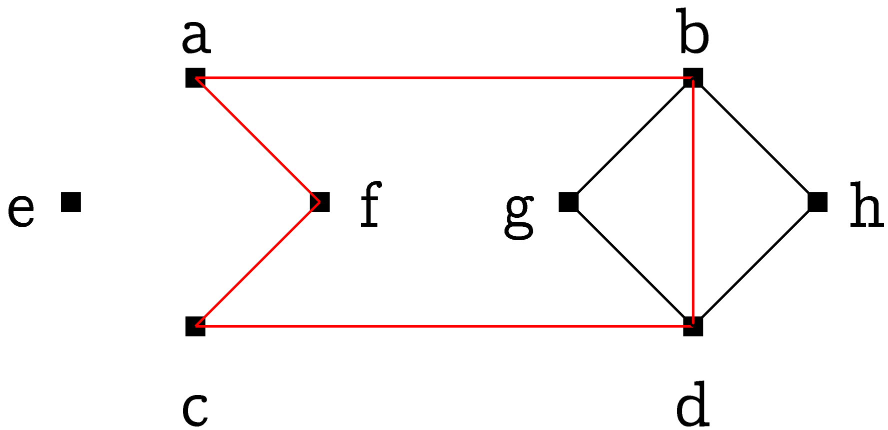
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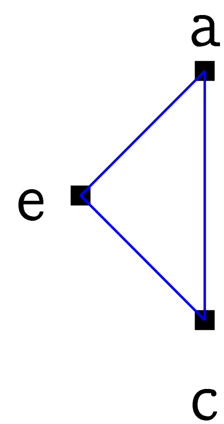
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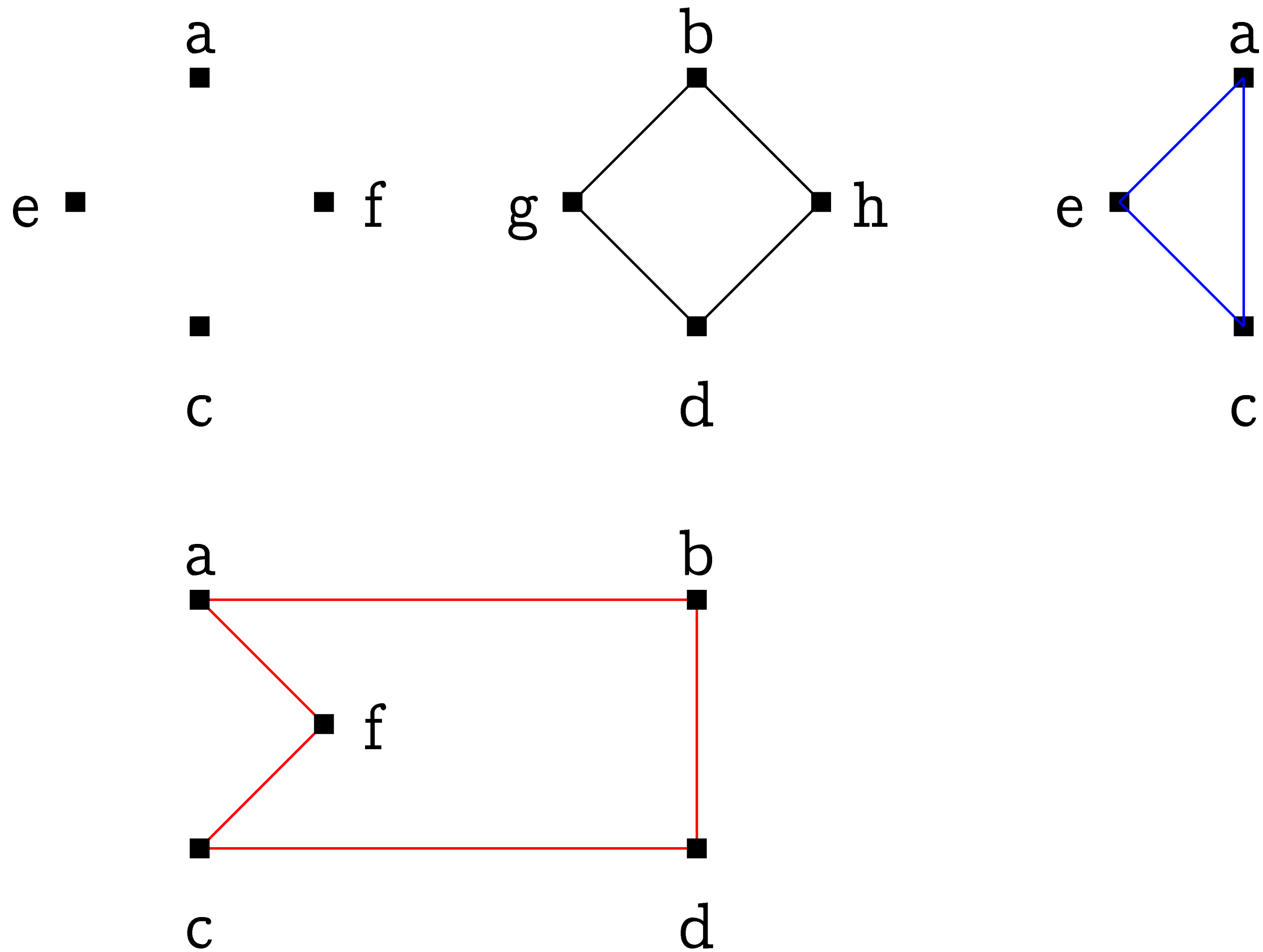
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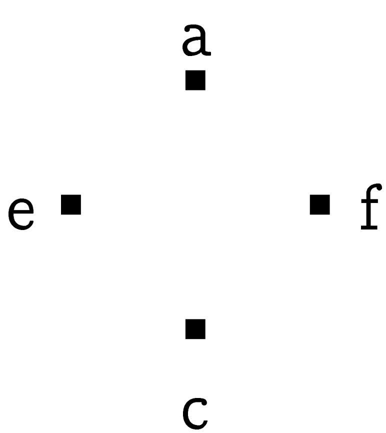
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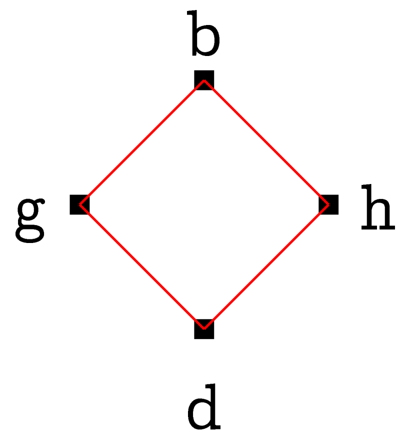
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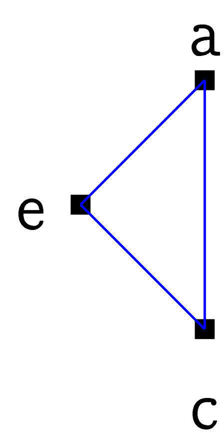
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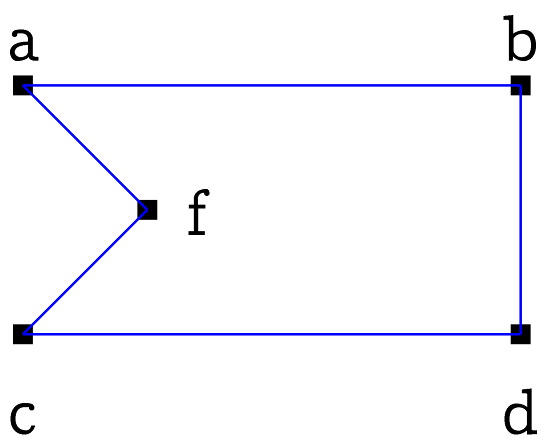
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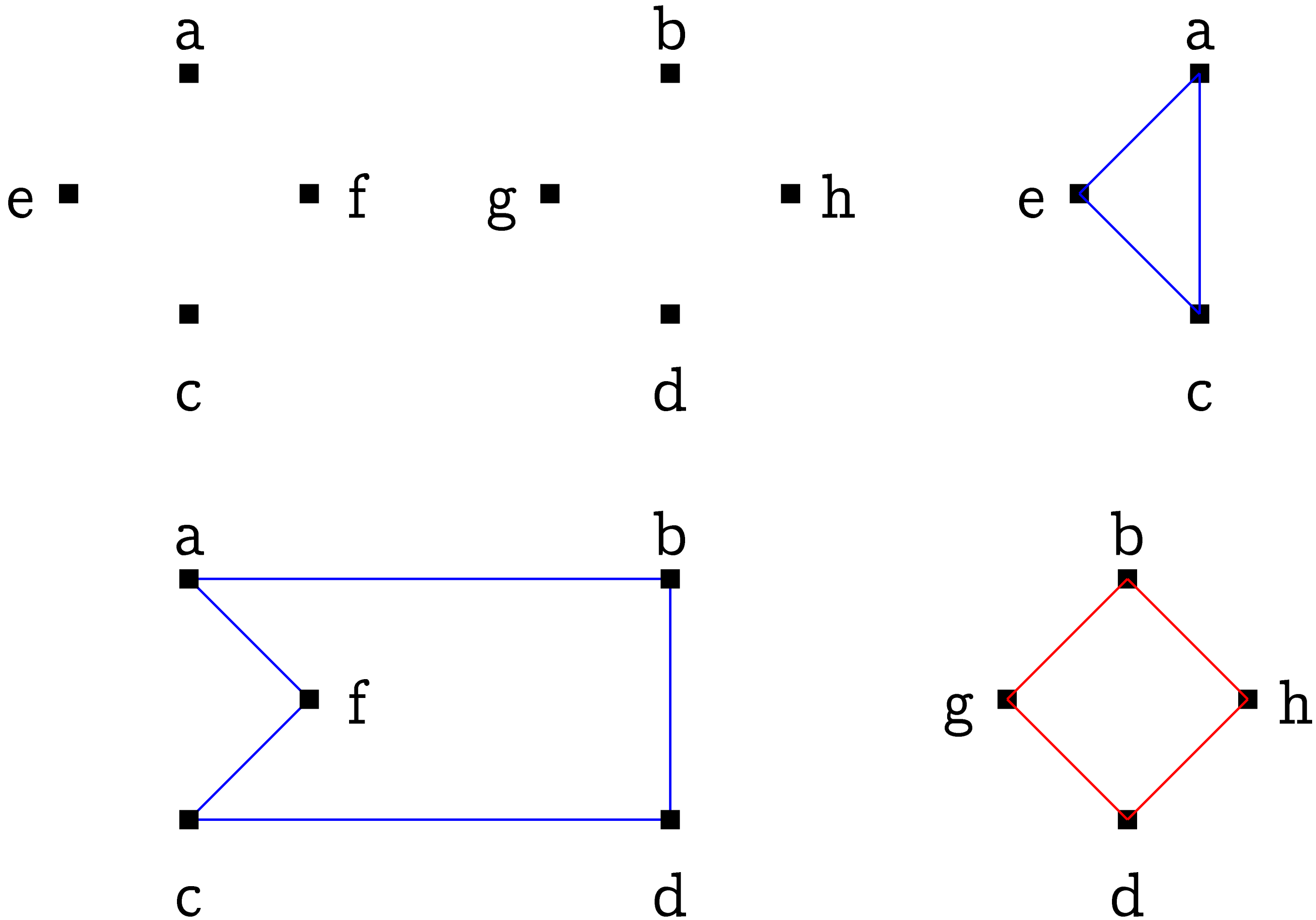

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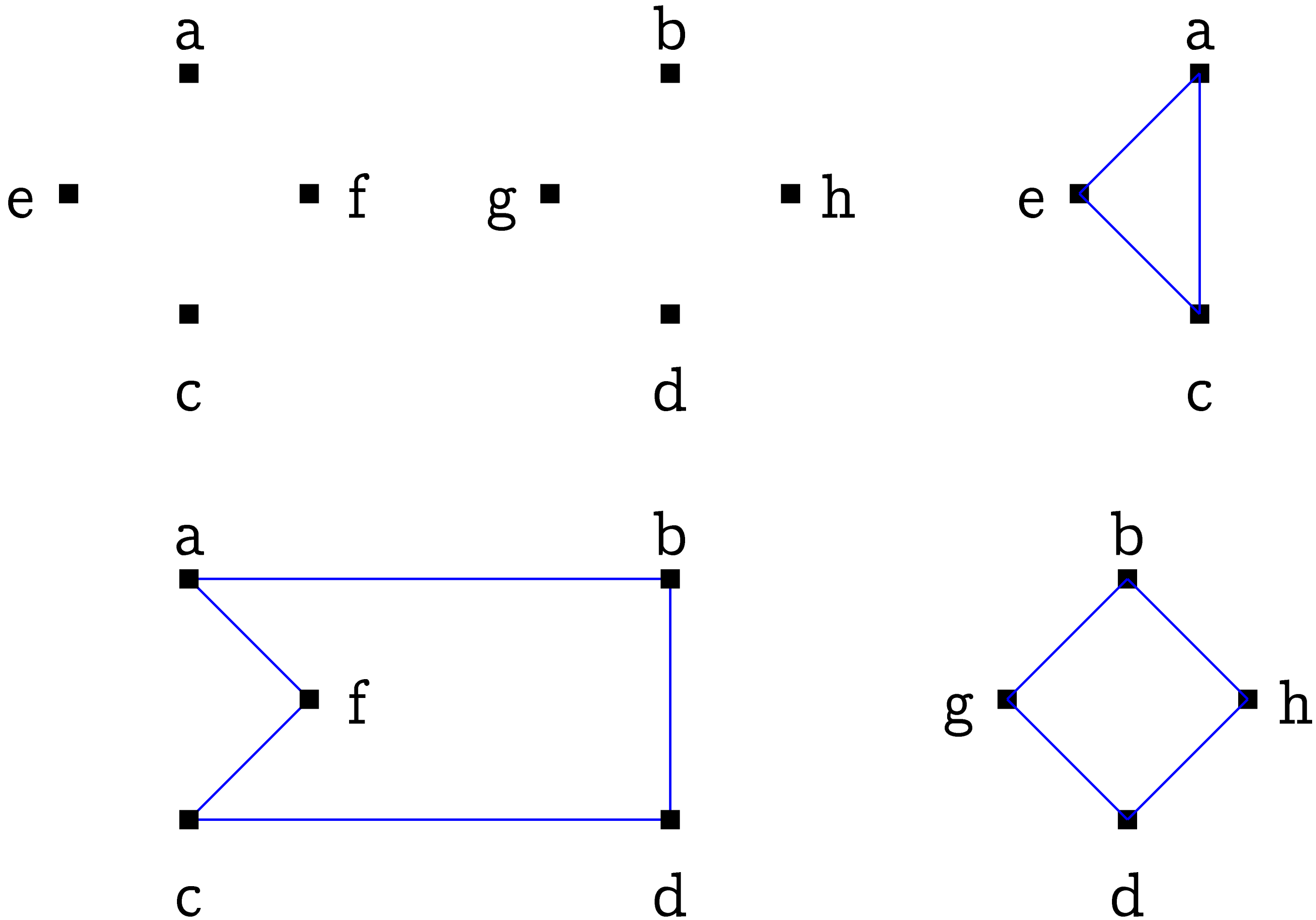
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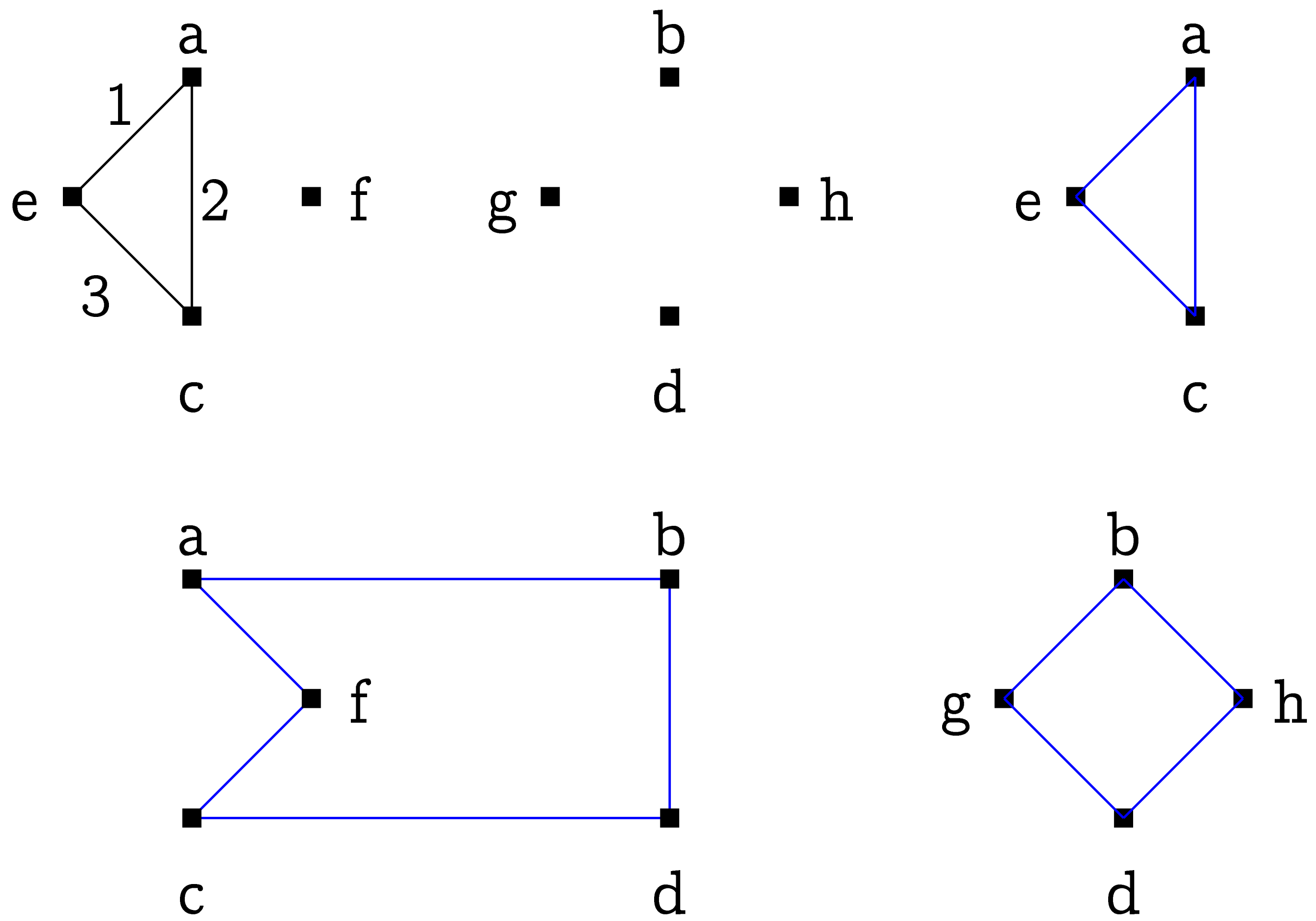
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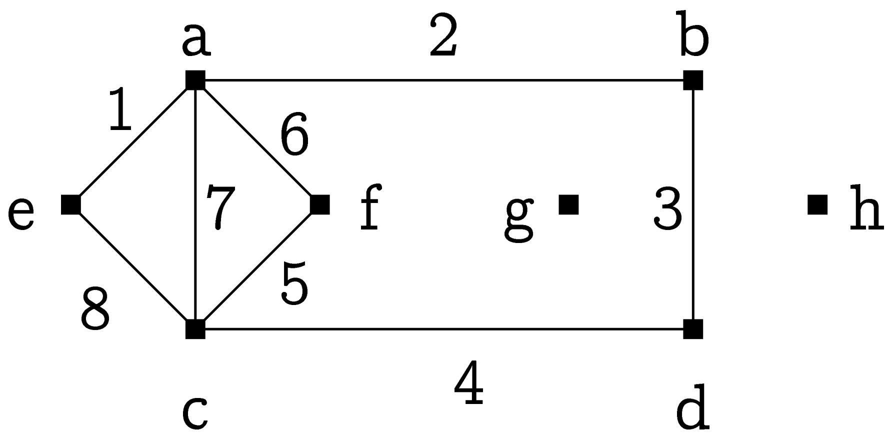
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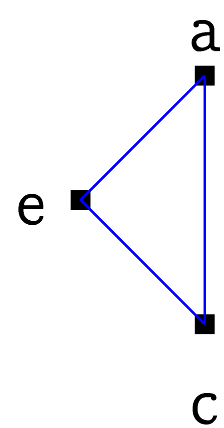
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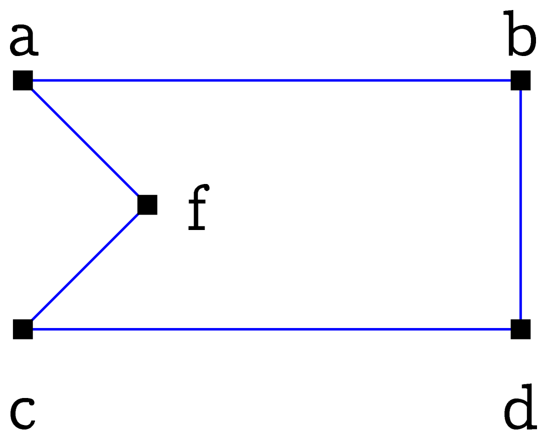
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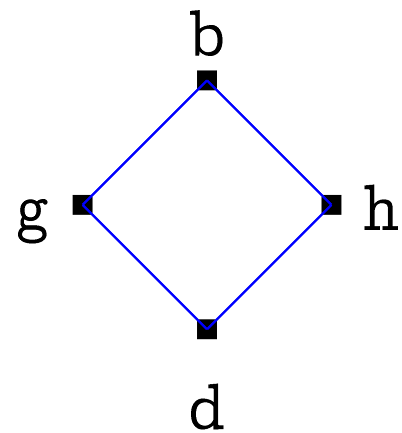
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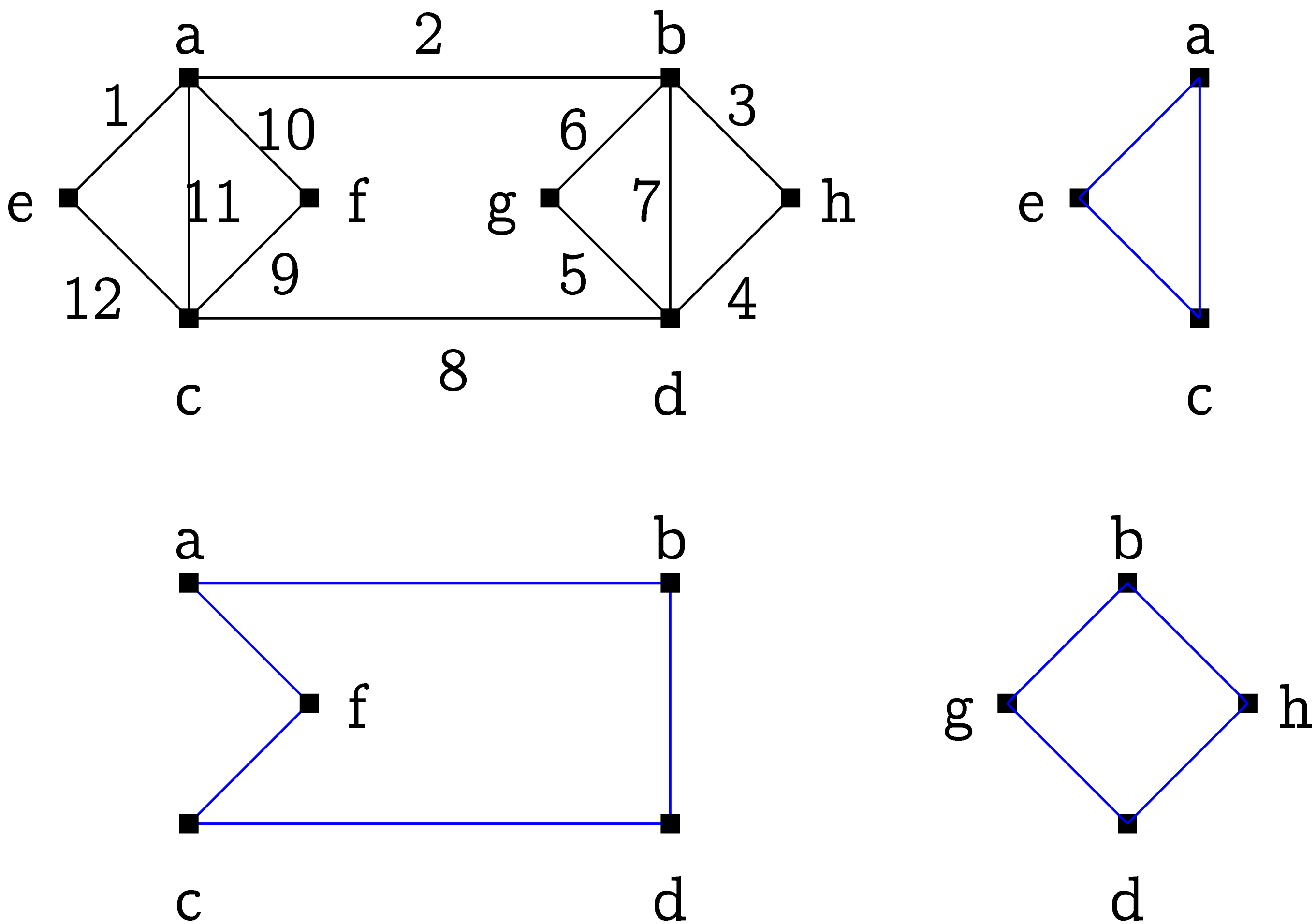
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Corollary. Connected graph G is semi-Eulerian $\Leftrightarrow$ the graph G has exactly two vertices with odd degree.

Proof $\Rightarrow$. Let $x \overset{P}{\rightsquigarrow} y$ be a walk in G covering each of the edges of G exactly once.

Add an edge e to G so that $\mathcal{E}(e) = \{x, y\}$.

The graph we obtain is Eulerian ($x \overset{P}{\rightsquigarrow} y \overset{e}{\text{———}} x$ is a Eulerian walk), thus all the vertex degrees are even.

Hence in the original graph x and y have odd degree and all the other vertices have even degrees.

Proof $\Leftarrow$. Let x and y be the two vertices of G having odd degree.

Add an edge e to G so that $\mathcal{E}(e) = \{x, y\}$.

As a result, all the vertex degrees become even, thus there exists a Eulerian walk P.

W.l.o.g assume that the last edge in this walk is e. Removing it from P we obtain the required walk. $\square$

The proof gives an algorithm for finding such a walk:

> Add an additional edge e, find the Eulerian walk and then drop e from it.

Fleury's algorithm for finding a Eulerian walk in Eulerian graph $G = (V, E)$:

1. Pick any vertex $u \in V$ as the first one in the walk. Let $i := 0$ and $v_0 := u$.

2. Pick an edge e incident with vertex v_i, add it to the walk and delete it from the graph G. Let v_{i+1} be the other endpoint of e and let $i := i + 1$.

   - If e is a bridge, pick it *only* if there is no other alternative.

3. Repeat the last step until all the edges are deleted.

Theorem. Fleury's algorithm is correct (i.e. it will always run successfully and produce a Eulerian walk).

Proof. The algorithm produces some walk P starting from u. At some point it stops, because it reaches a vertex v_n, that has all the incident edges deleted. Considering the vertex degrees, it is obvious that $v_n = u$.

We have to show that at that moment all the edges are deleted.

Let G_i be the graph remaining of G after step i. Then $G_0 = G$ and G_{i+1} contains one edge less than the graph G_i. Let H_i be the connected component of G_i containing the vertex u.

Note that the degrees of all the vertices of G_i (except for, possibly, u and v_i) are even. If $u = v_i$ then also $\deg(u)$ is even. If $u \neq v_i$ then $\deg(u)$ and $\deg(v_i)$ are odd.

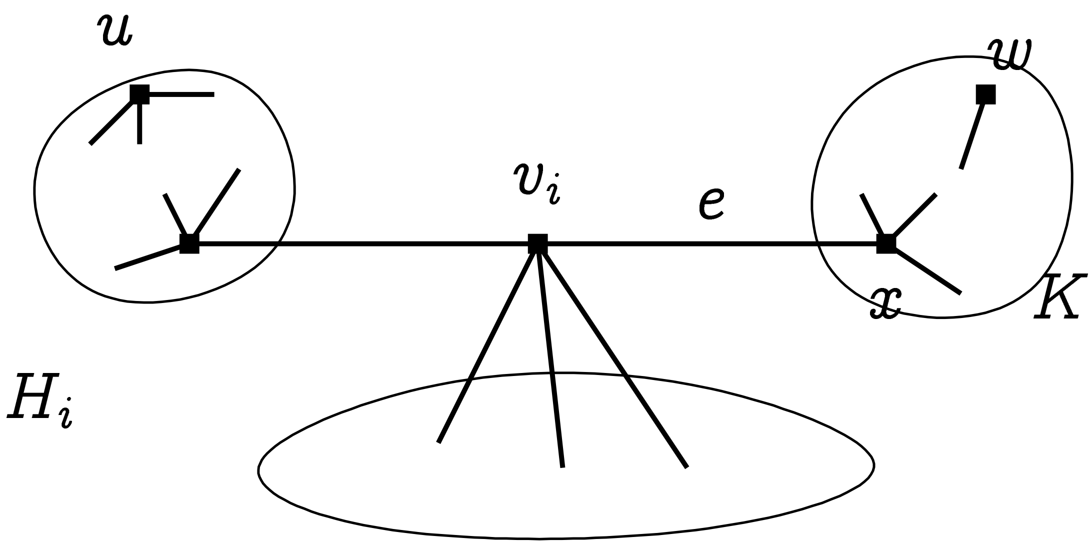
We will show that all the remaining connected components of G_i are isolated vertices.

We will use induction over i. If $i = 0$ then $G_0 = G = H_0$, and G_0 has only one connected component, thus the claim holds.

Let the claim hold for G_i. Consider first the case $u \neq v_i$. In order to give the proof for G_{i+1}, it is enough to prove that there is at most one bridge incident with v_i in the graph G_i.

- If so, then we are done, because the connected components of G_{i+1} are the following.
  - If we deleted a non-bridge, the connected components did not change.
  - If we deleted a bridge, it was the last edge incident with v_i. The component H_i is divided into two new components – v and $H_{i+1} = H_i \backslash v$. The first one is an isolated vertex, the second one contains vertex u.

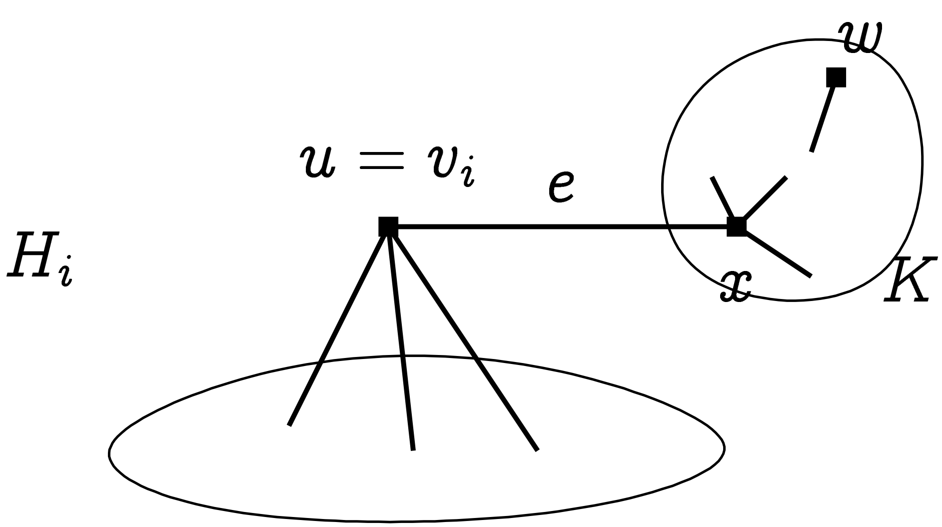
If at least two bridges were incident to v_i then:



- There exists an edge e incident to v_i such that the connected component of $H_i - e$ not containing v_i does not contain u either.
- $\deg_{H_i}(x)$ is even. Thus $\deg_K(x)$ is odd.
- There has to exist another vertex w of K so that $\deg_K(w)$ is odd. At the same time, $\deg_K(w) = \deg_{H_i}(w)$ an this had to be even.

If $u = v_i$, it is enough to show that there are no bridges incident with u, i.e. G_i and G_{i+1} have the same connected components.

If u would have an incident bridge,



there would again exist a vertex w with odd degree. $\square$