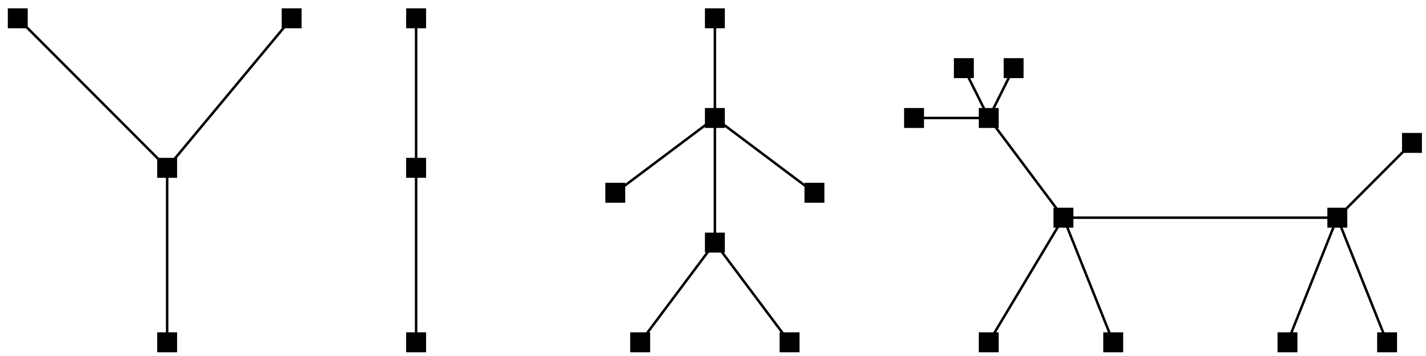
Trees

A graph that has no cycles is called a *forest*.

A forest with one connected component is caleld a *tree*.



A tree vertex with degree 1 is called a *leaf*.

Proposition. All trees are bipartite.

Proof. Start dividing the vertices alternatively into two sets starting from some vertex and moving along the edges. We can not get a contradiction, since there are no cycles.
□

Proposition. Let G be a graph with n vertices, m edges and k connected components. Then $n - k \leq m$.

Proof. Induction over m.

If $m = 0$, then each vertex of G is a separate connected component, i.e. $k = n$. The inequality holds.

Let $m > 0$. Removing an edge from graph G, we obtain a graph with $m - 1$ edges. There are two possibilities:

- The number of connected components did not increase. Induction hypothesis gives $n - k \leq m - 1$. Thus $n - k \leq m$ as well.
- The number of connected components increased by one. Induction hypothesis gives $n - (k + 1) \leq m - 1$. Thus we also have $n - k \leq m$. □

Theorem. Let $T = (V, E)$ be a graph with n vertices. Any two of the following claims imply the third.

(i). T is connected.

(ii). T has no cycles.

(iii). T has $n - 1$ edges.

This theorem gives two alternative definitions of a tree.

Proof.

(i) & (ii) $\Rightarrow$ (iii). Induction over n.

If T has one vertex, then all the edges of T are loops. But loops are cycles, which are prohibited by (ii). Thus T must have $0 = 1 - 1$ edges.

Let T have n vertices.

T has no cycles $\Longrightarrow$ T has a vertex v with degree 0 or 1.

> **Theorem.** Graph with all vertex degrees ≥ 2 has a cycle.

T is connected $\Longrightarrow$ the defree of v is not 0.

The subgraph T' induced by $V \backslash \{v\}$ is connected and has no loops, hence by the induction hypothesis it has $n - 2$ edges.

It remains to note that T has one more edge than T'.

(ii) & (iii) $\Rightarrow$ (i). Assume that T is not connected.

Let $T_1, \dots, T_k$ be the connected components of graph T. They are all connected and cycle-free, thus according to the proof (i) & (ii) $\Rightarrow$ (iii) the number of edges is one less than the number of vertices in all of them.

Alltogether, graph T has $n - k$ edges. Since T has $n - 1$ edges by (iii), we must have $k = 1$, hence T is connected.

(i) & (iii) $\Rightarrow$ (ii). Assume T has a cycle. Removing one edge form the cycle, we get a connected graph with n vertices and $n - 2$ edges, contradiction with the proposition proven earlier. $\square$

Intermezzo: mathematical induction

Theorem. Graph T is a tree iff it is connected and all of its edges are bridges.

Proof. $\Rightarrow$ Let T have n vertices and $n-1$ edges. Consider an edge. If we remove it, we are left with a graph having n vertices and $n-2$ edges, thus it can not be connected according to the first proposition. Thus this edge was a bridge.

$\Leftarrow$ If T had a cycle, then all of the edges of this cycle would be non-bridges. Thus T can not have cycles and, being connected, it is a tree. $\square$

Teoreem. Let T be a graph with n vertices. The following claims are equivalent.

1. T is a tree.
2. Between any two vertices of T there is exactly one path.
3. T has no cycles, but adding an edge between any two vertices creates a cycle.

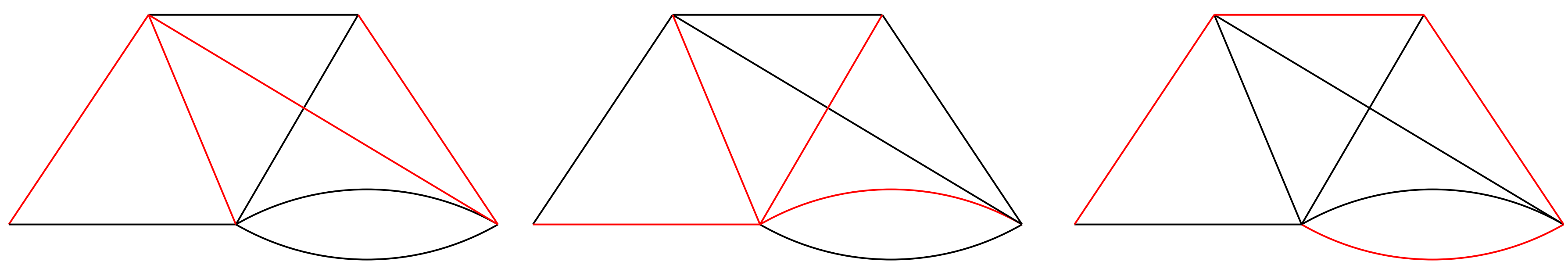
Proof. $1 \Rightarrow 2$. Between any two vertices there is at least one path – otherwise T would not be connected. If there were two different paths between two vertices, we would get a cycle and T would not be a tree.

$2 \Rightarrow 3$. T has no cycles, since otherwise we would get two different paths bewteen any two vertices on the cycle. Adding a new edge e between the vertices u and v, we obtain a cycle $u \rightsquigarrow v \overset{e}{\text{---}} u$.

$3 \Rightarrow 1$. Suppose T is not connected. When adding an edge between the vertices in different connected components we get no cycles, a contradiction with the assumption. $\square$

Spanning tree (aluspuu) of the connected graph $G = (V, E)$ is a such a subgraph T of G that their vertex sets coincide.

For a non-connected graph we can define the *spanning forest (alusmets)* which is the union of the spanning trees of ots connected components.



Let $G = (V, E)$ be a graph with n vertices and let us have a *weight* $w(e)$ defined for each of its edges $e \in E$.

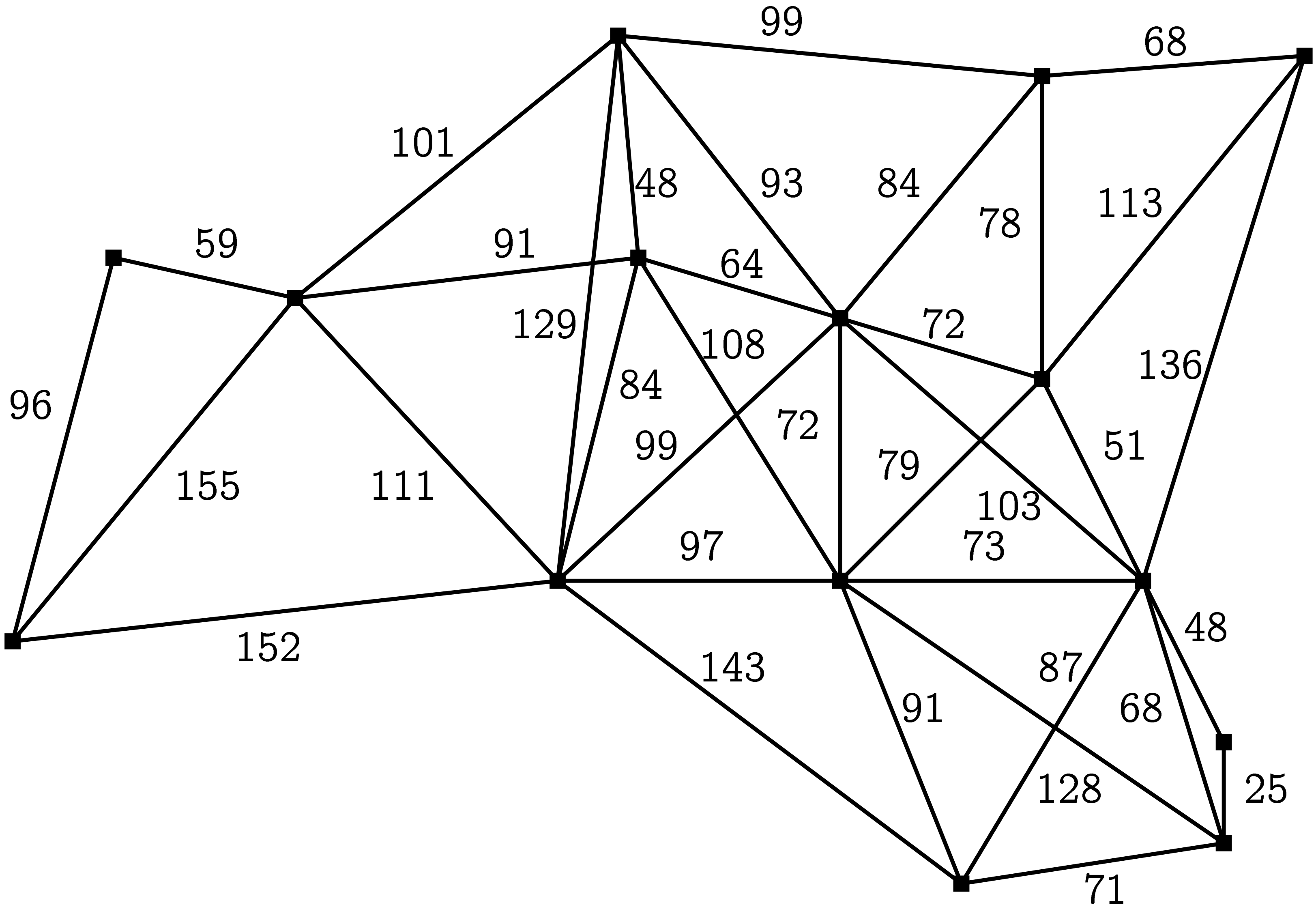
If $G' = (V', E')$ is a subgraph of G, then define $w(G') = \sum\limits_{e \in E'} w(e)$.

Algorithm (for finding the minimal weight spanning tree of G).

Select the edges $e_1, \ldots, e_{n-1}$ so that

- e_i differs from the edges $e_1, \ldots, e_{i-1}$;
- e_i does not form a cycle together with $e_1, \ldots, e_{i-1}$;
- e_i has the minimal weight among the edges satisfying the two conditions above.

Output $T = (V, \{e_1, \ldots, e_{n-1}\})$.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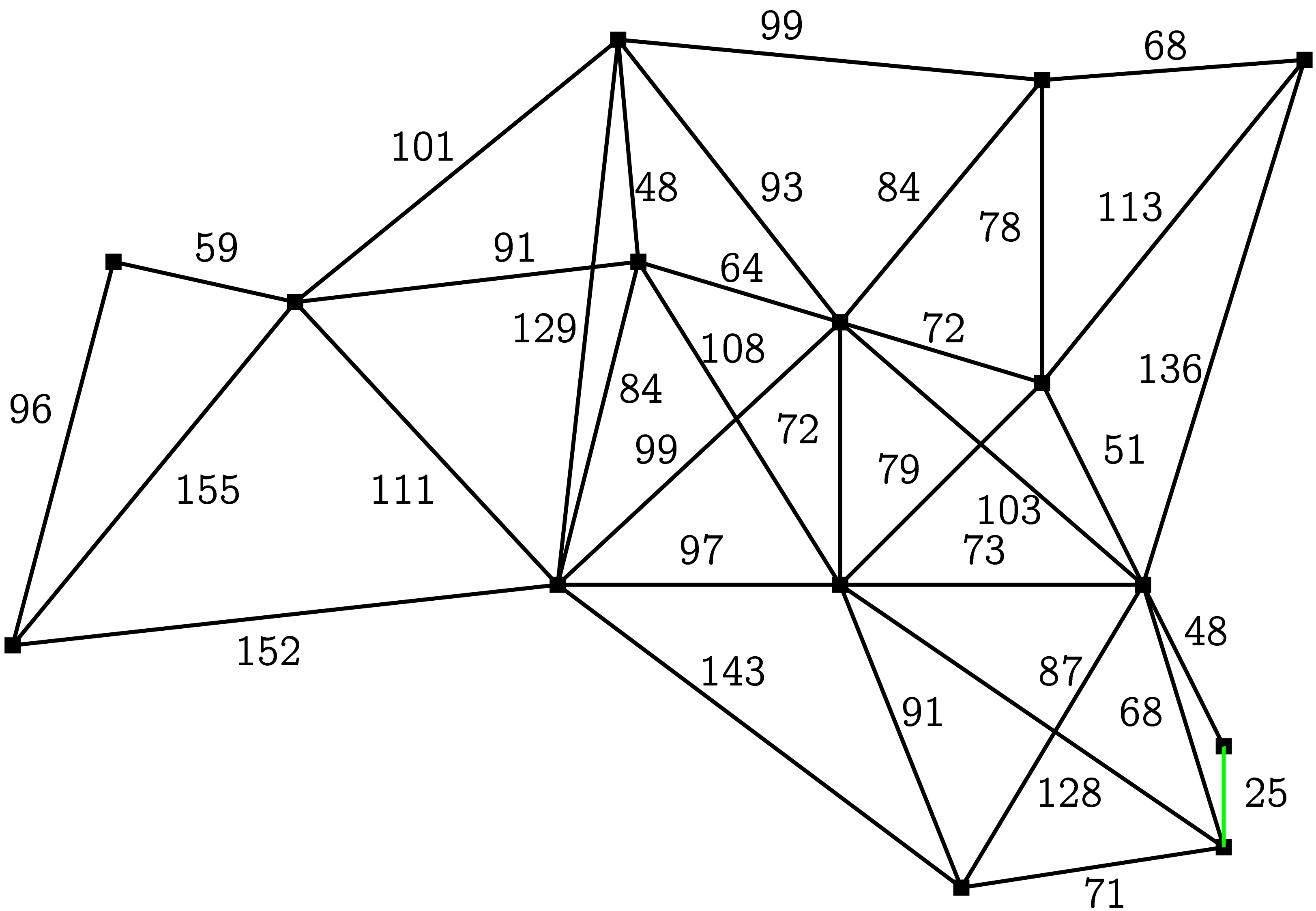
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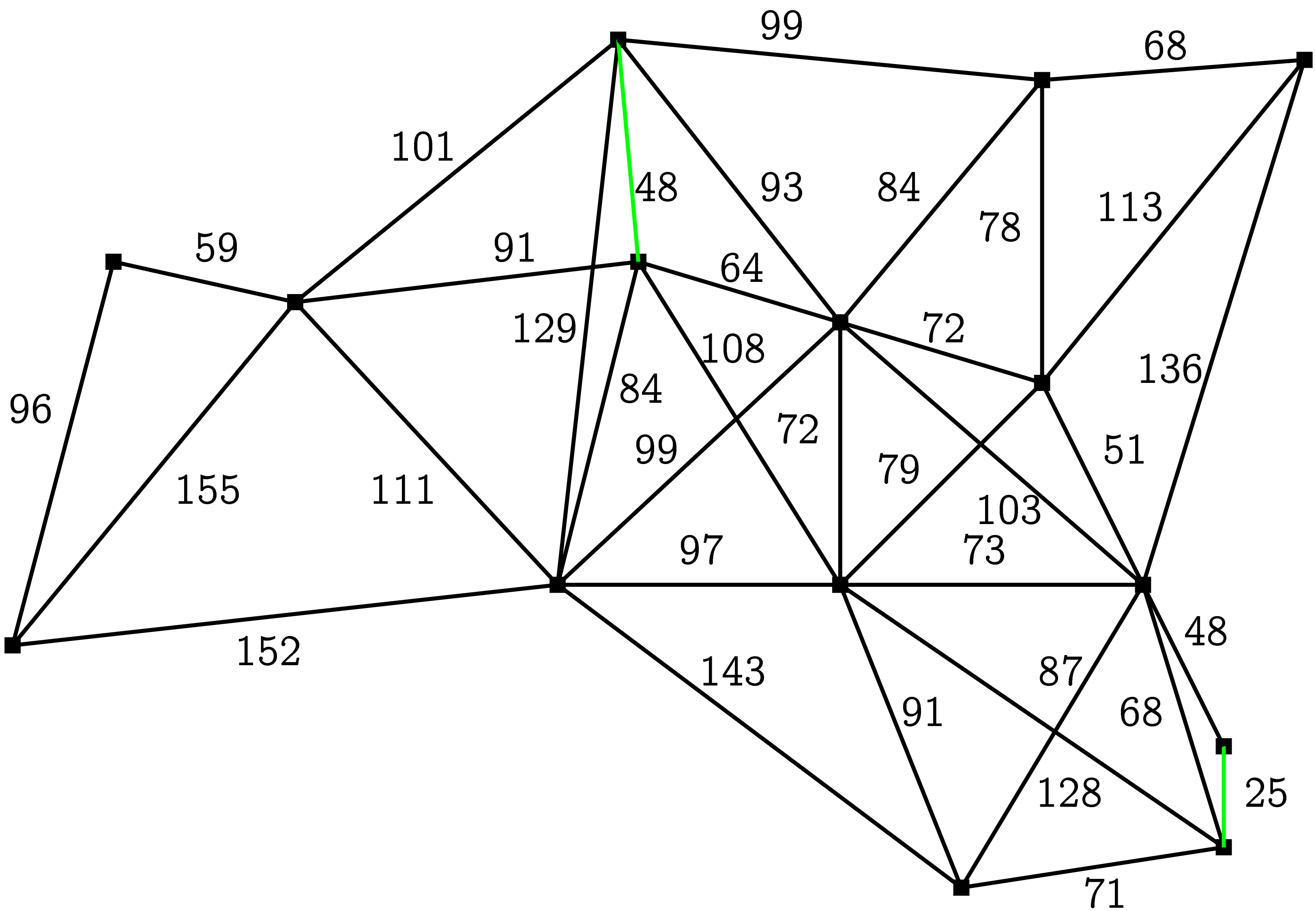
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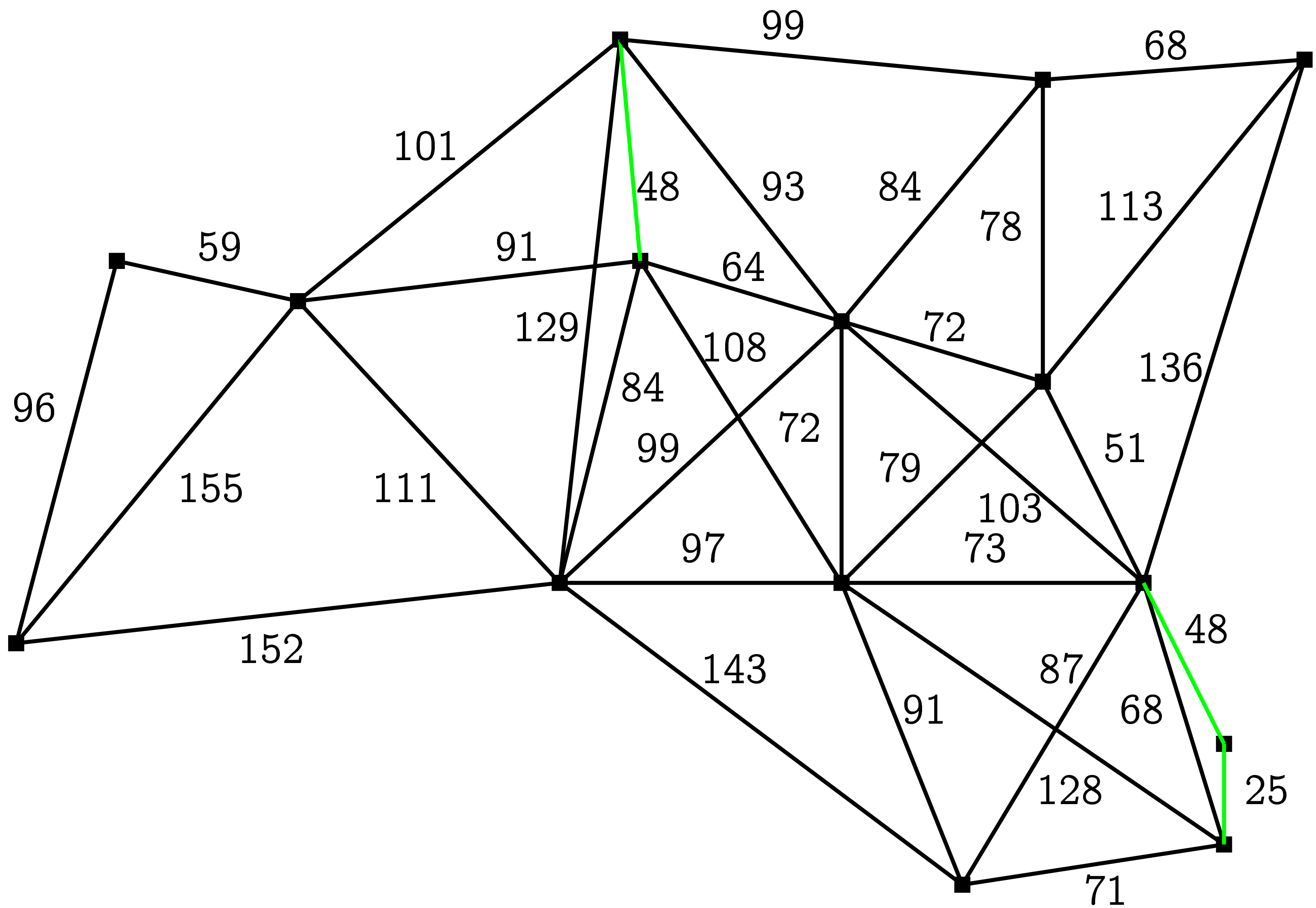
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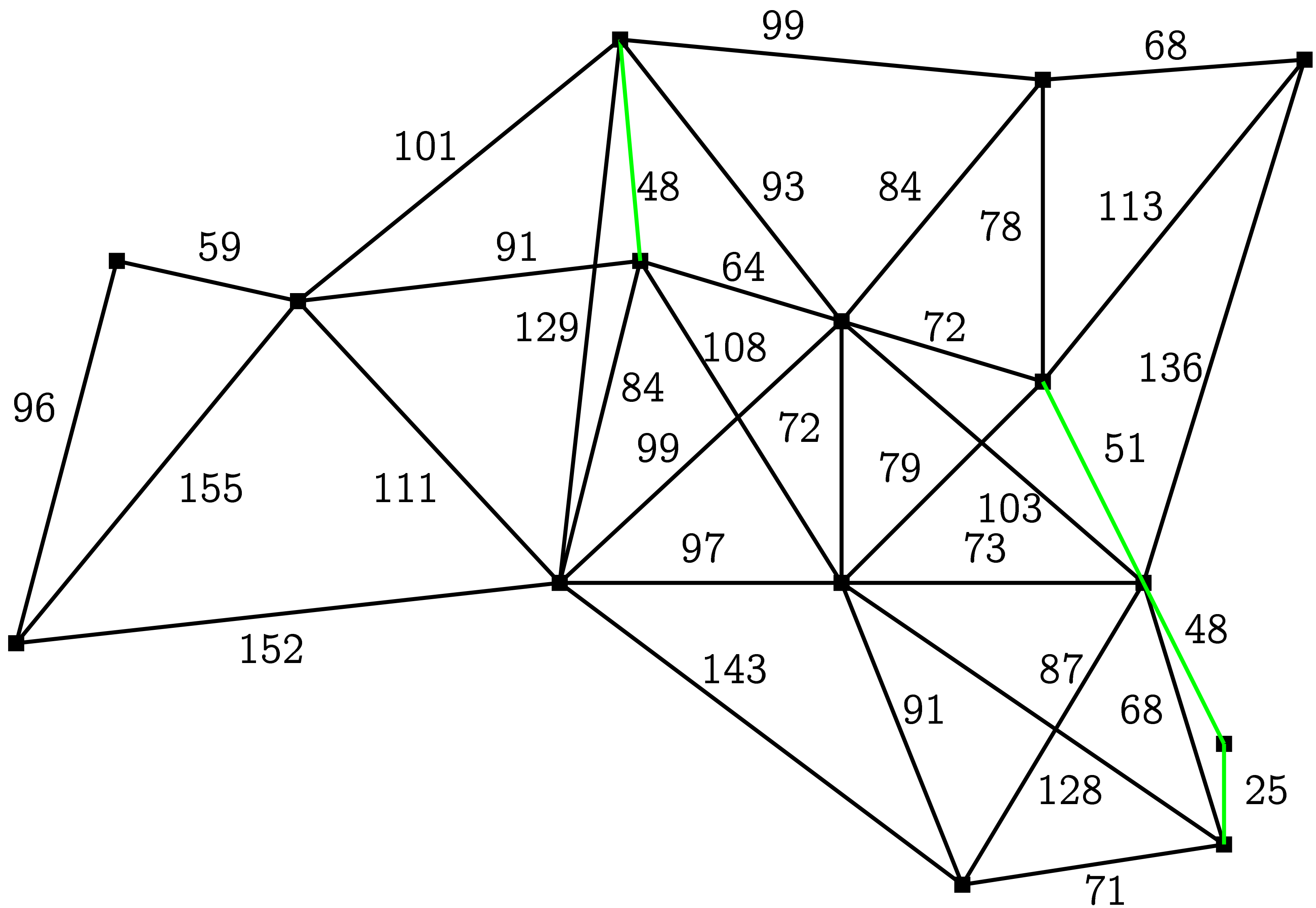
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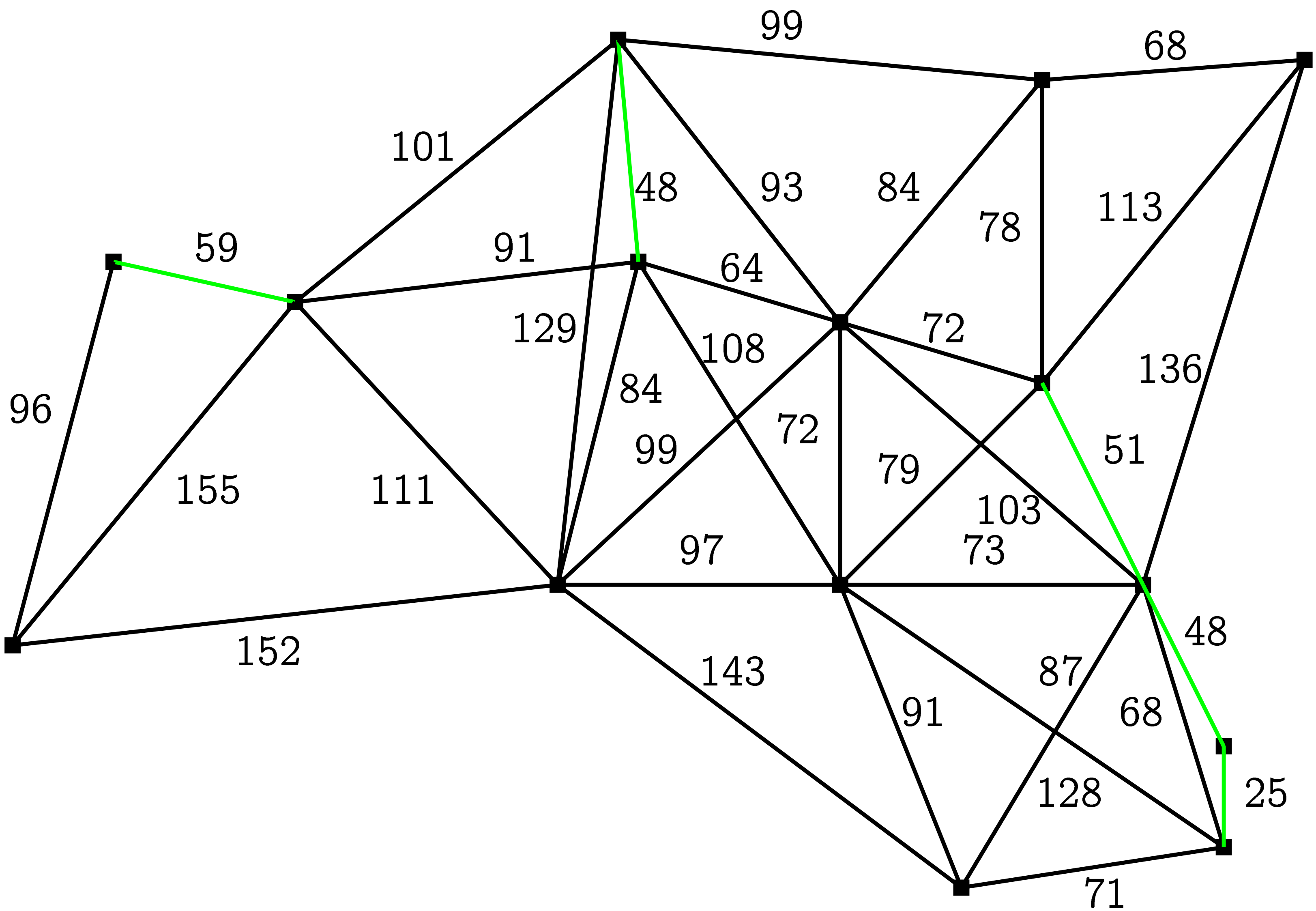
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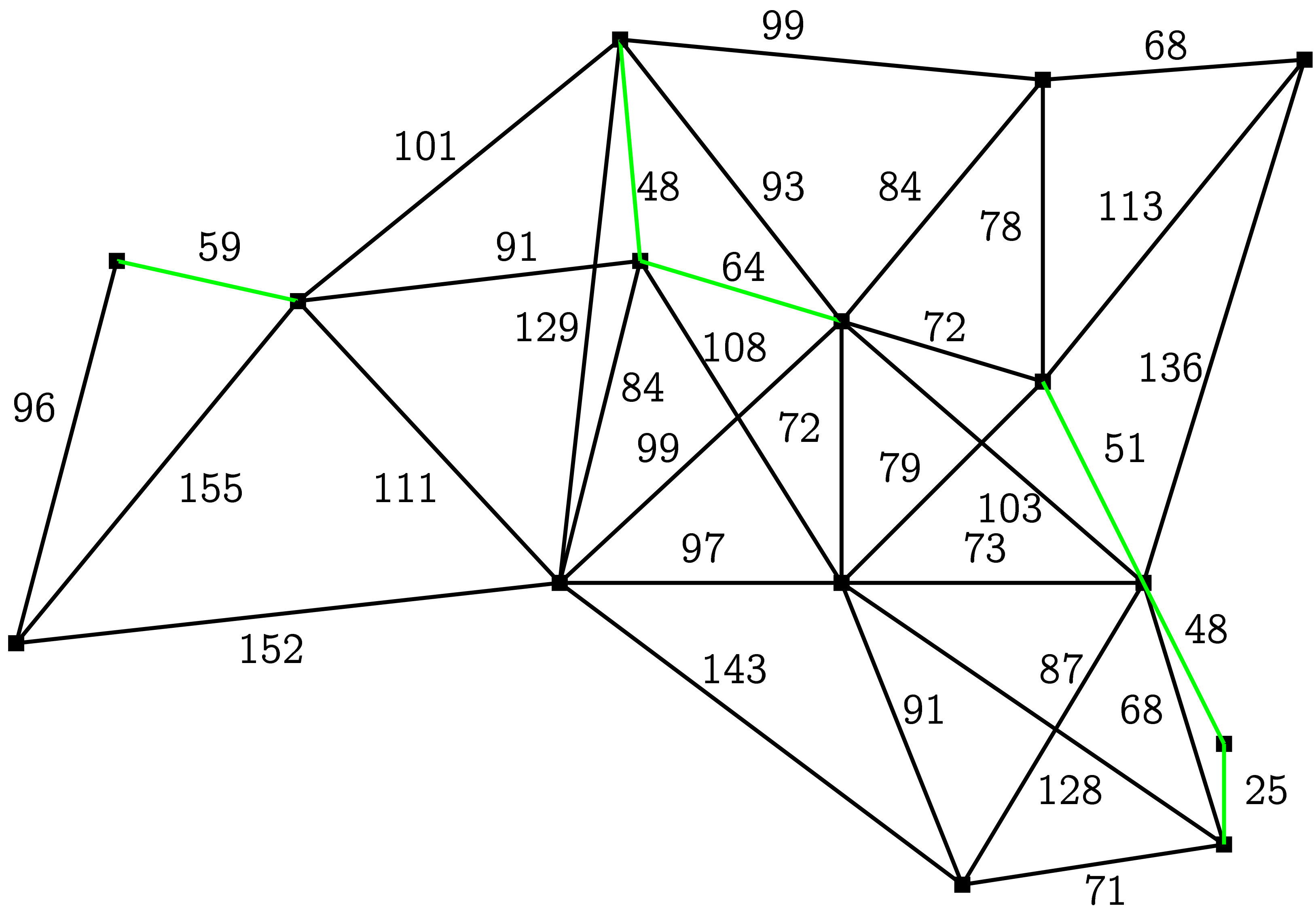
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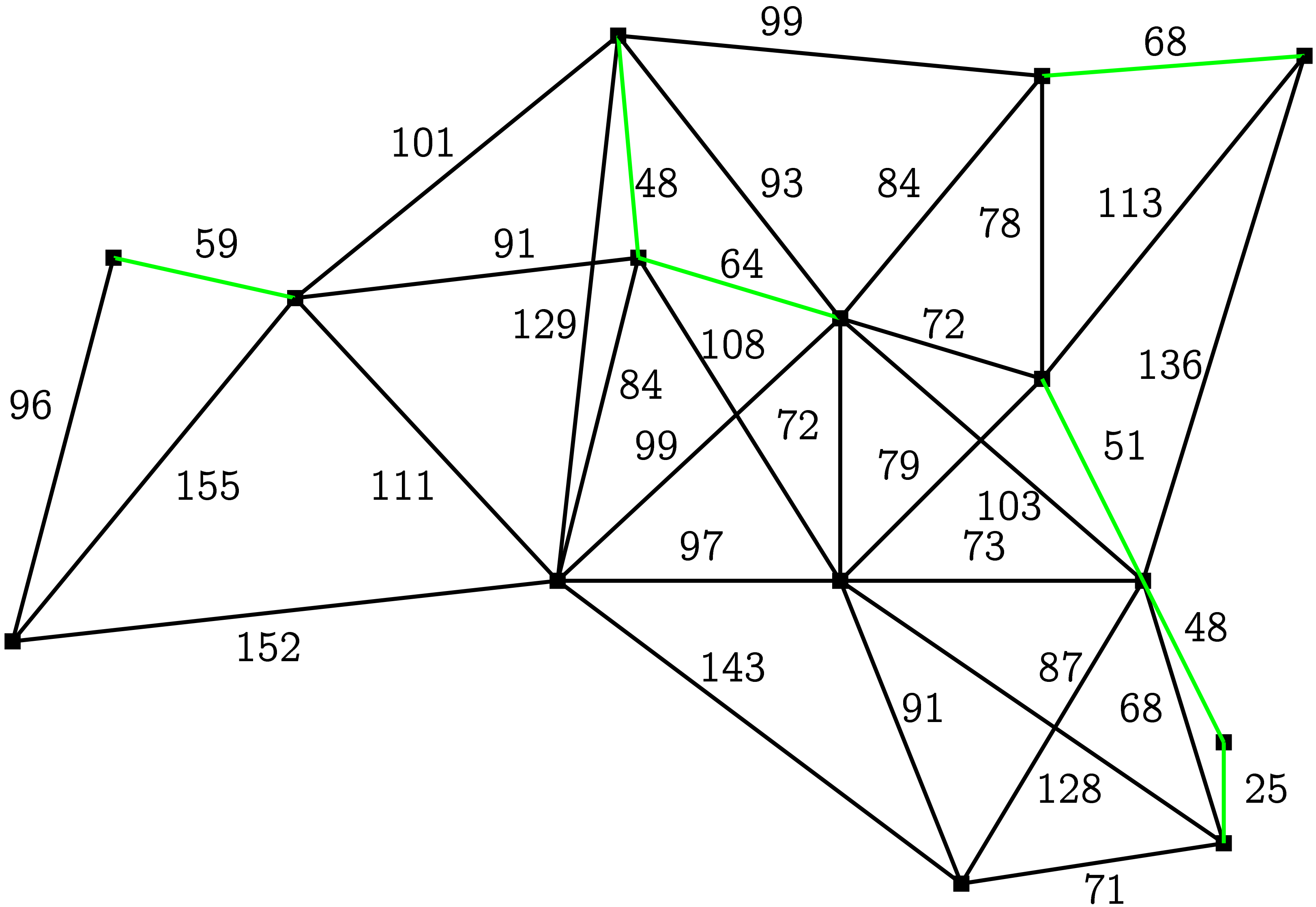
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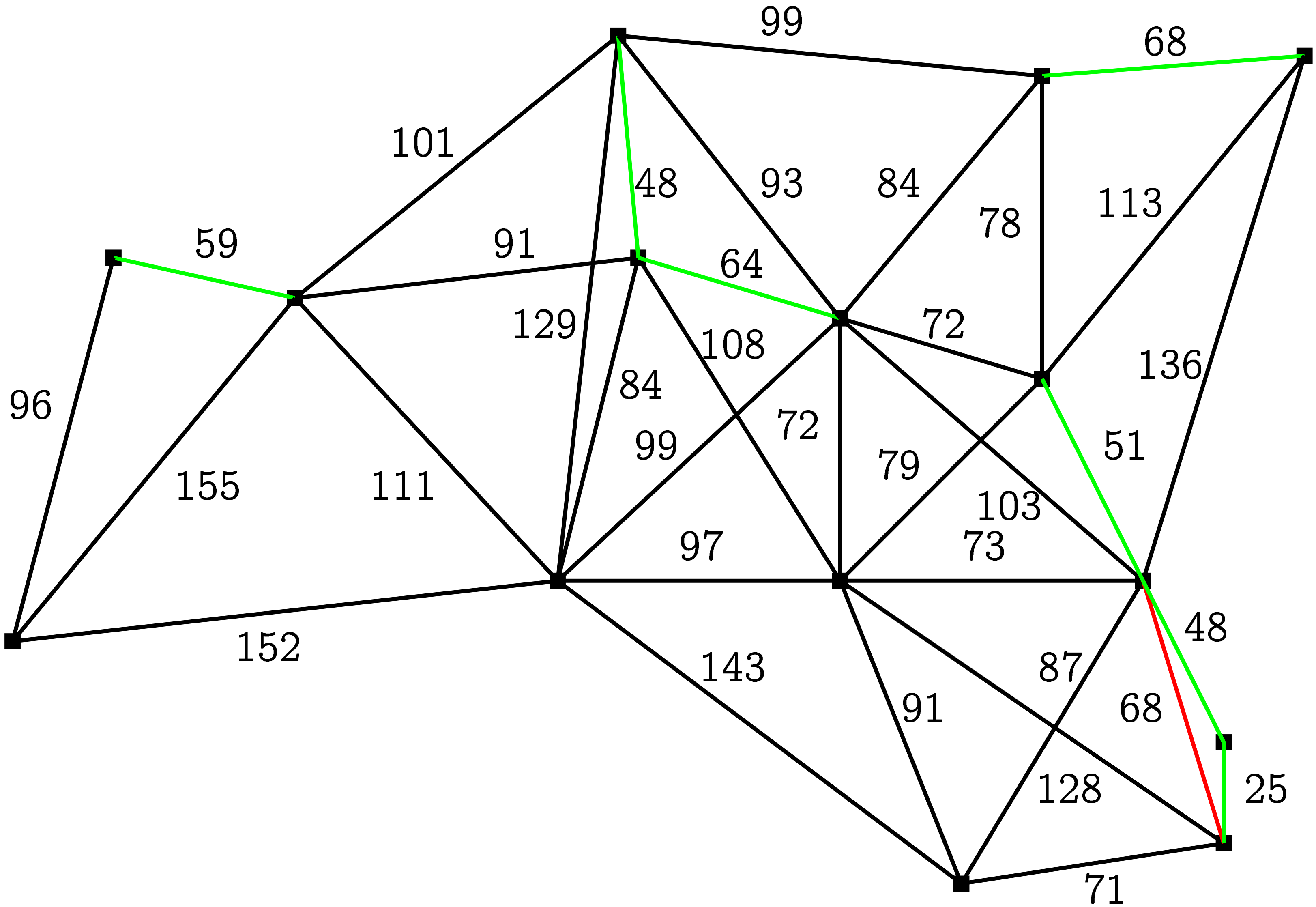
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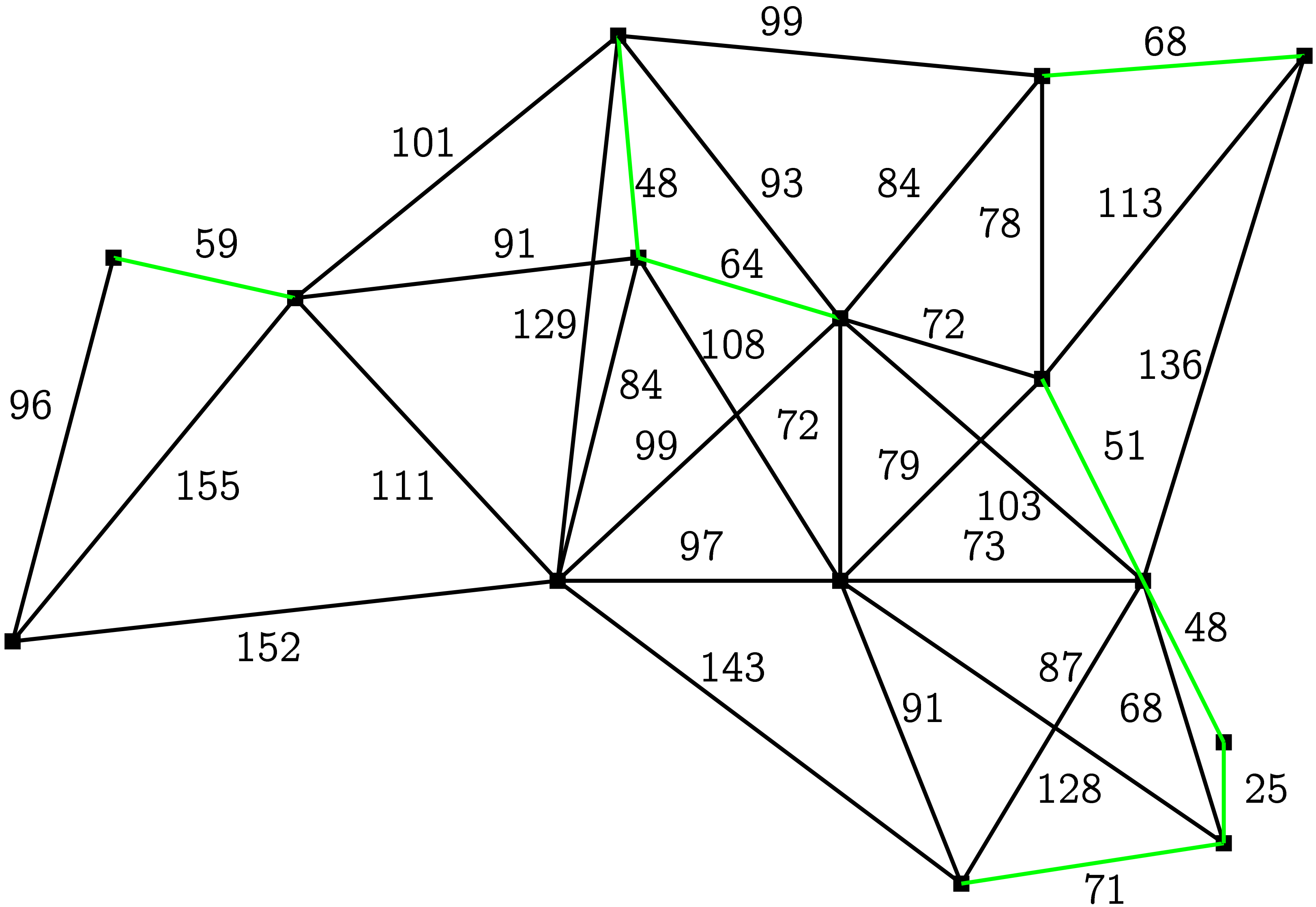
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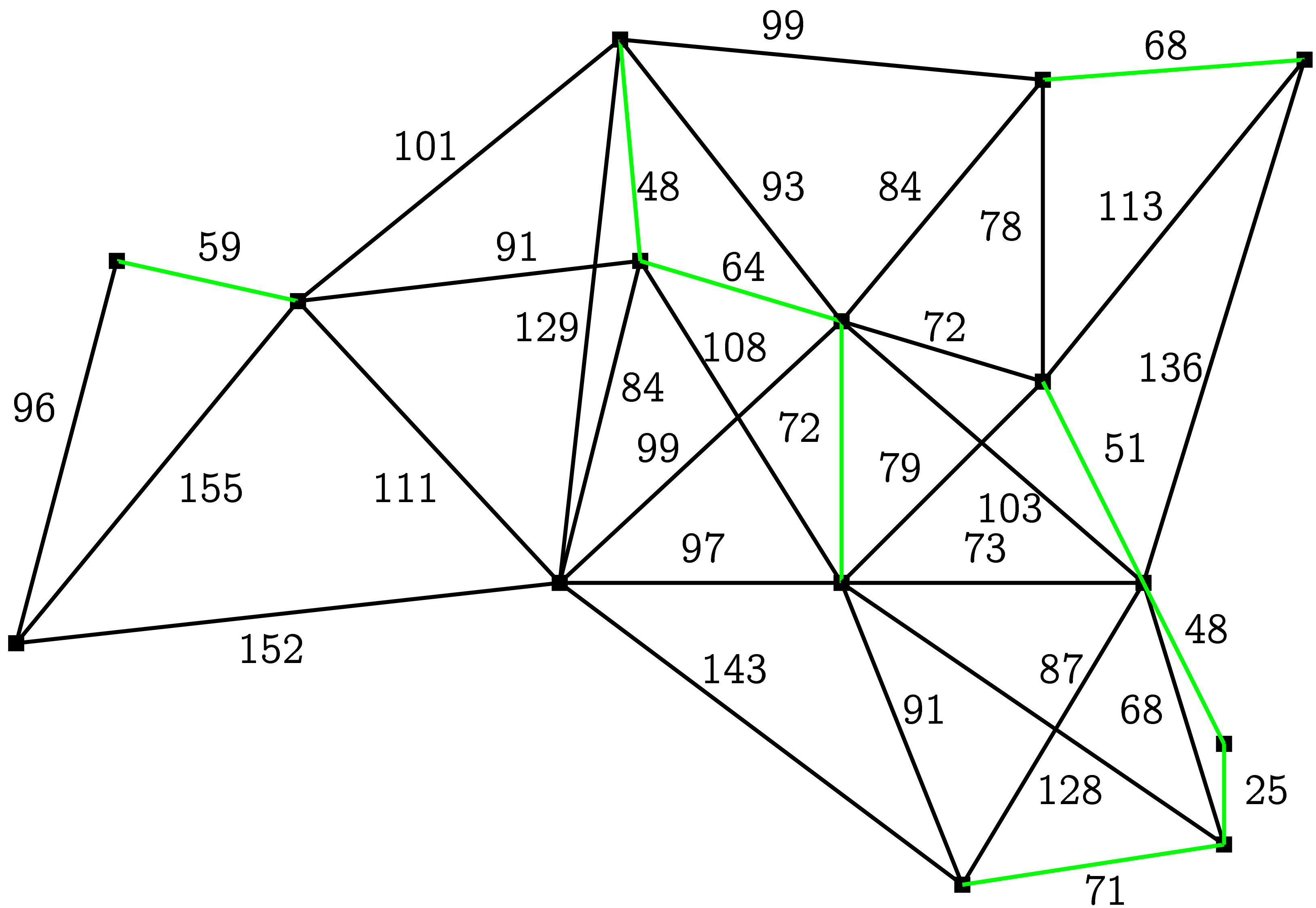
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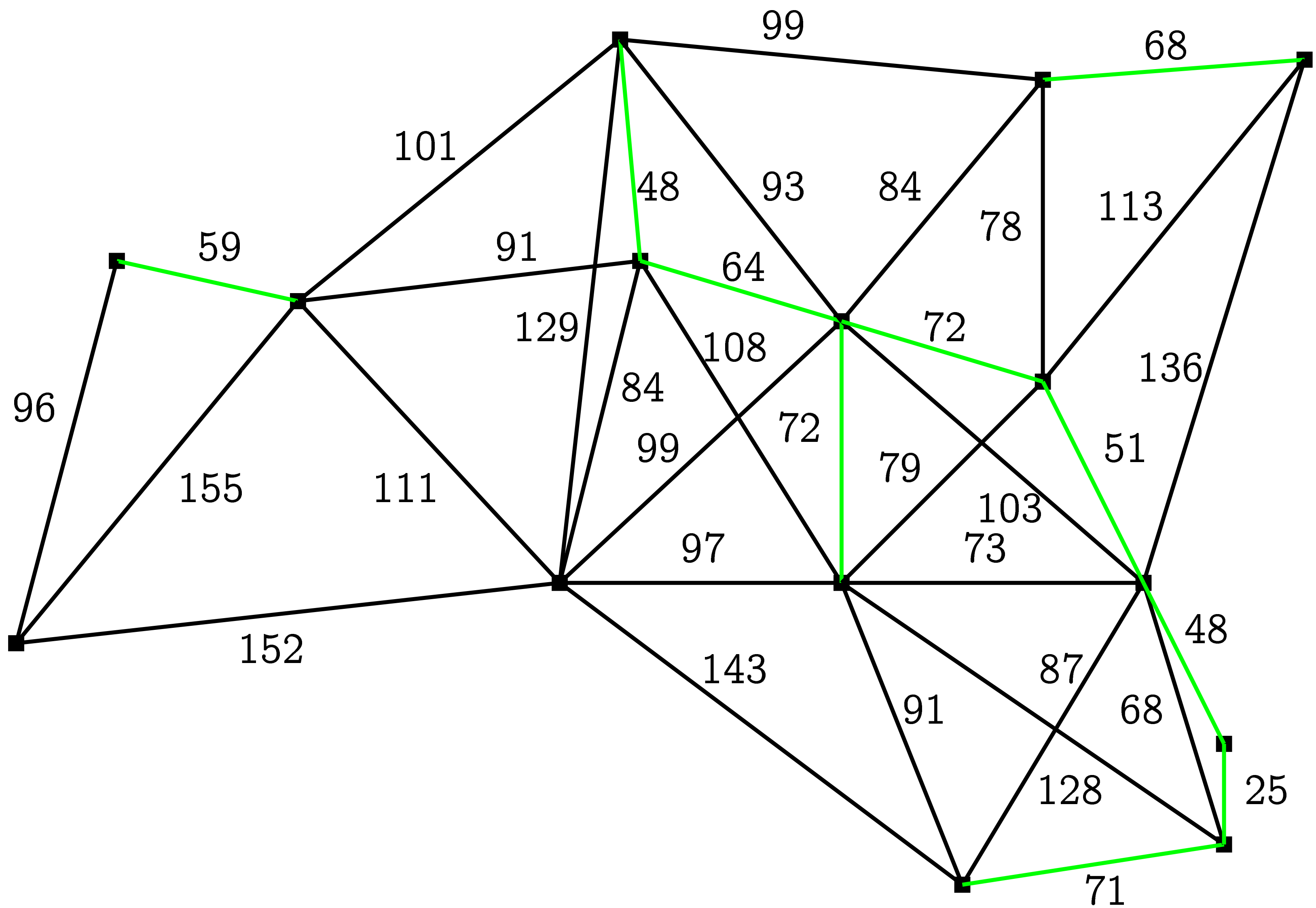
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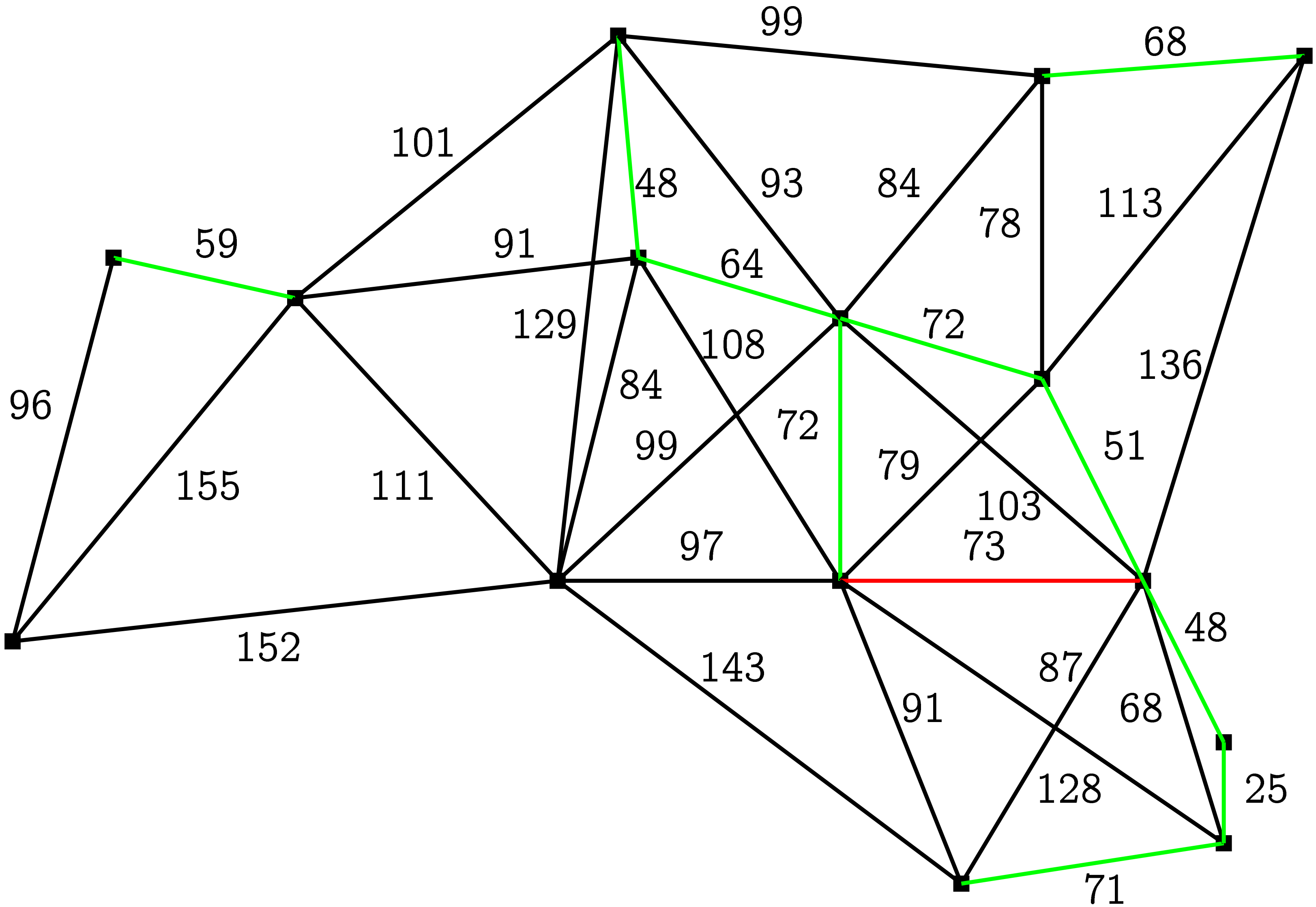
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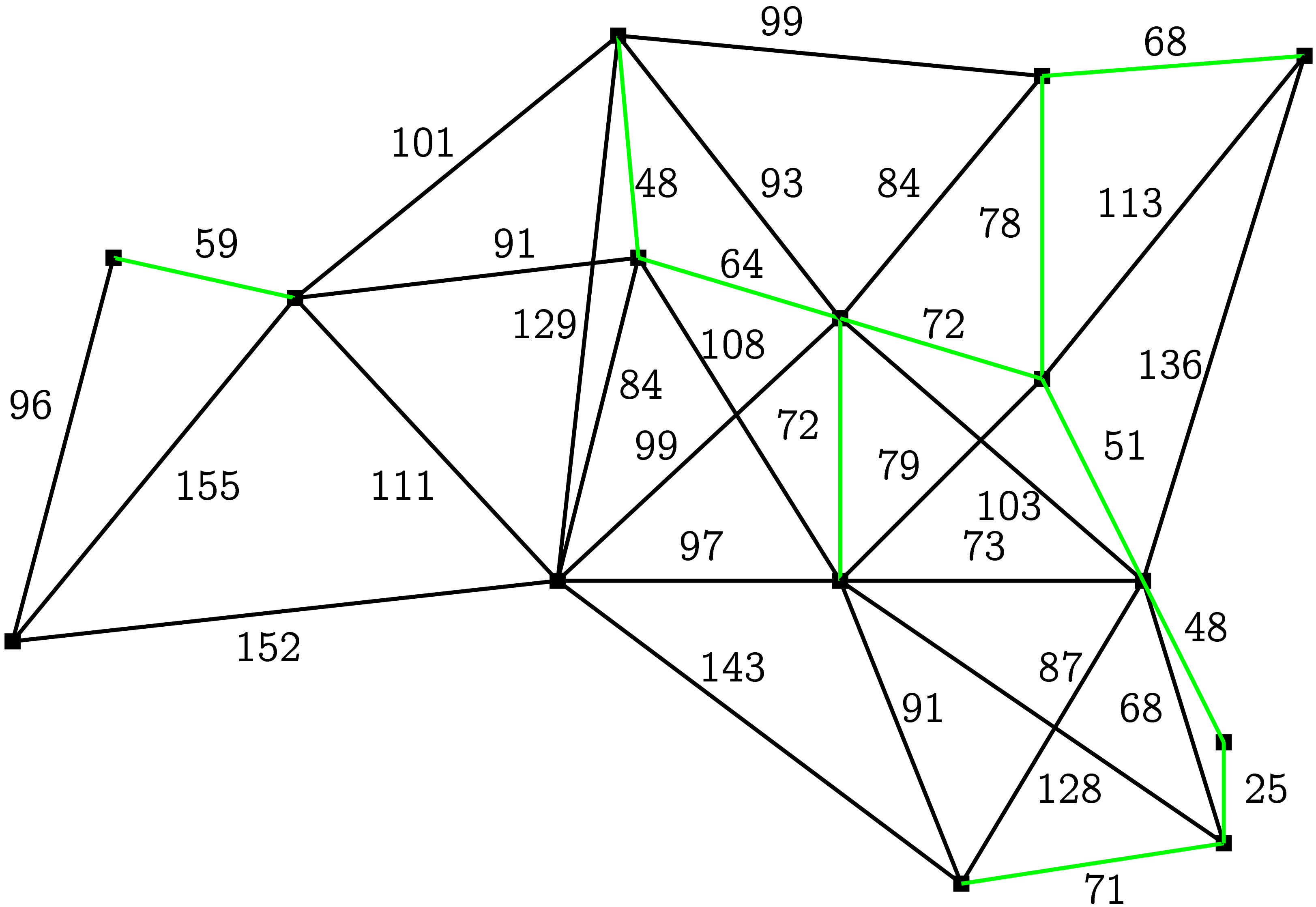
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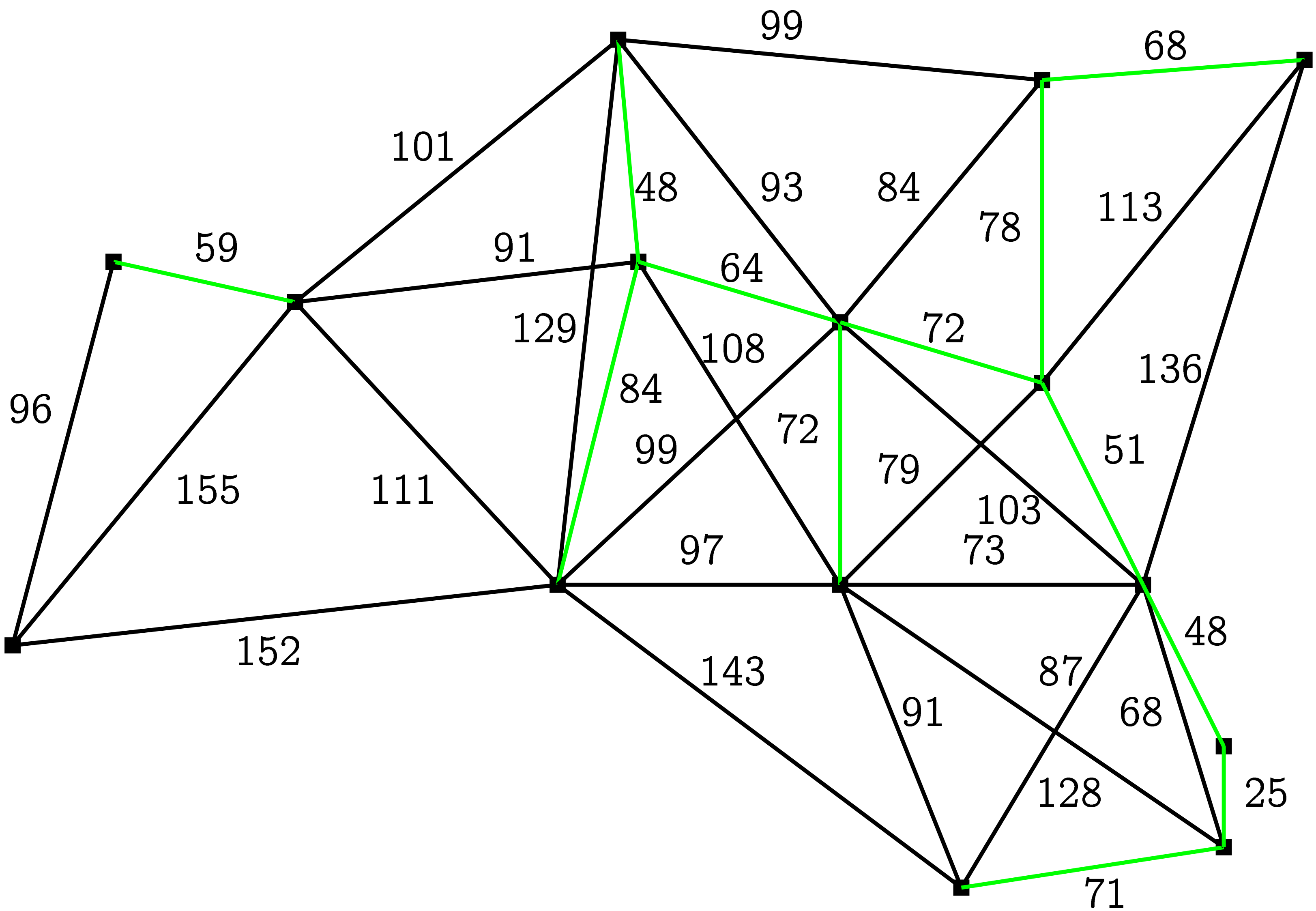
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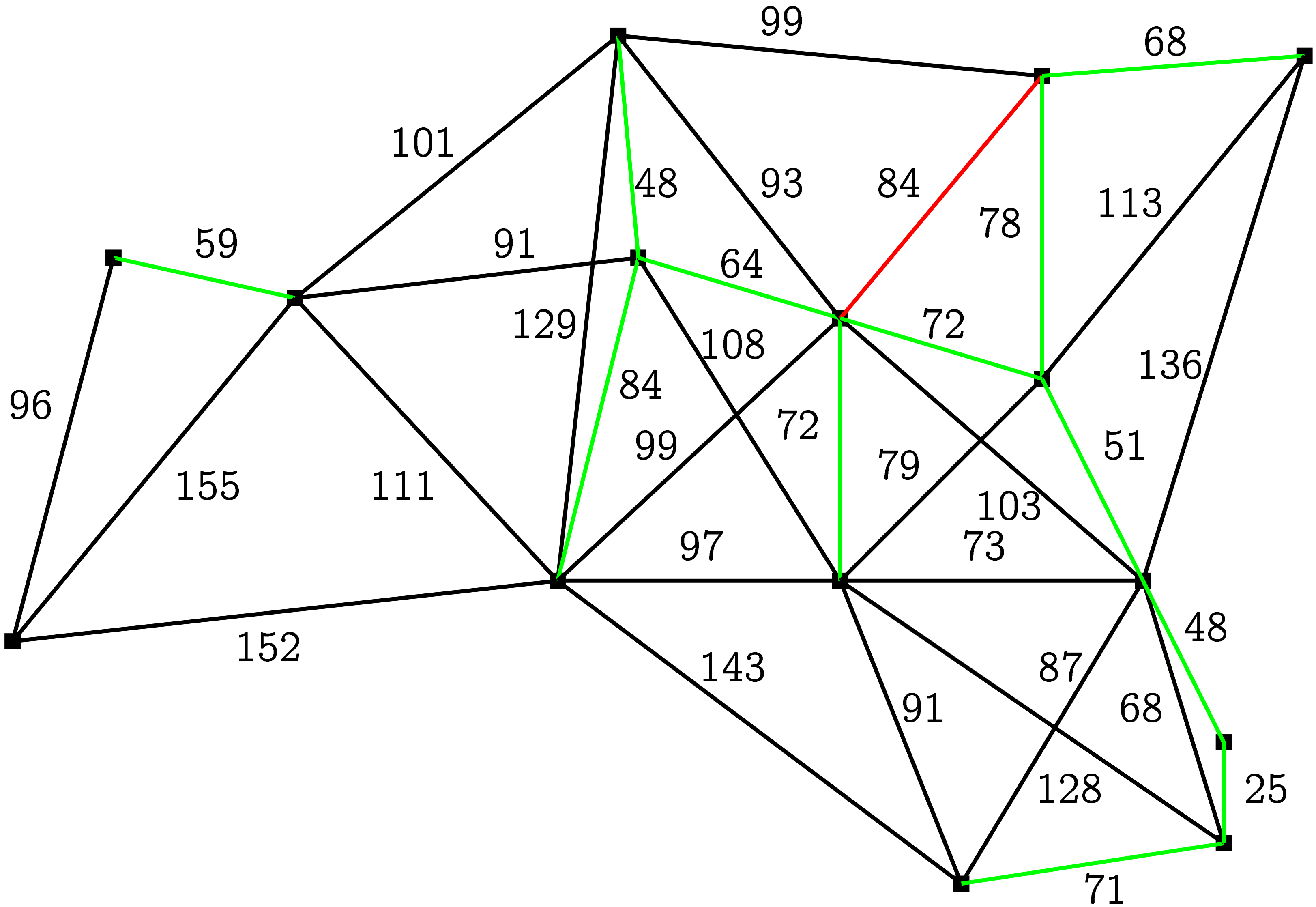
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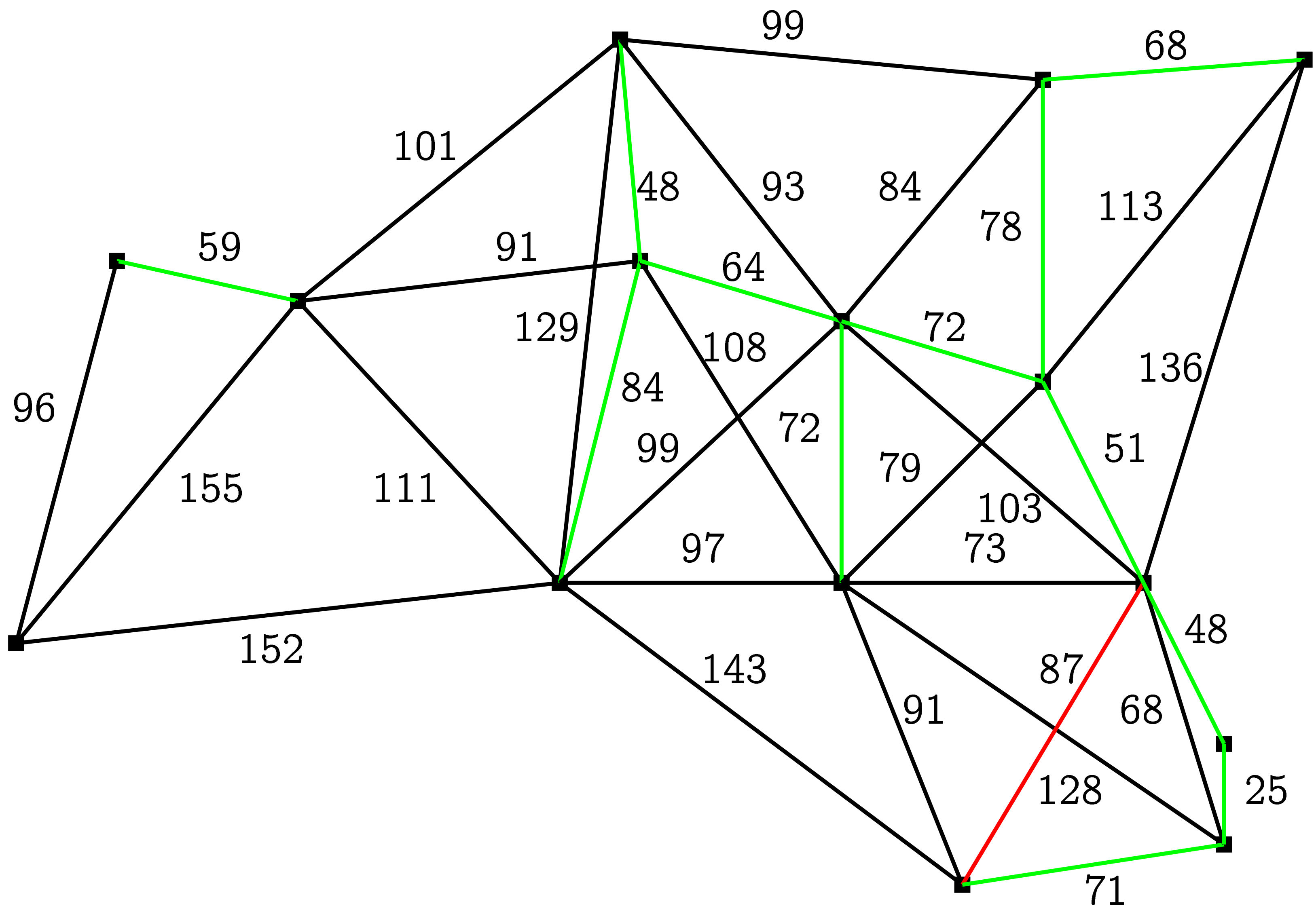
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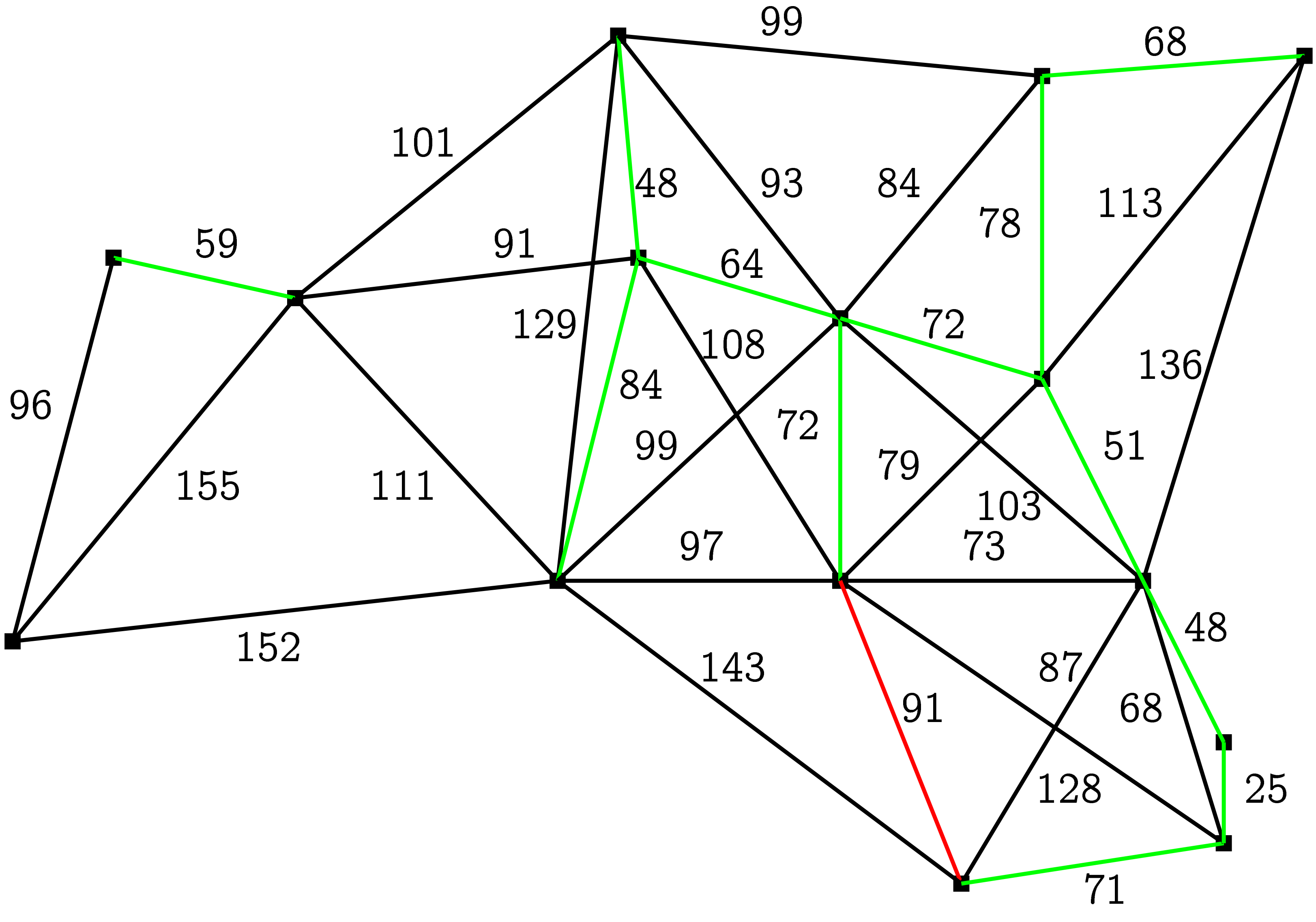
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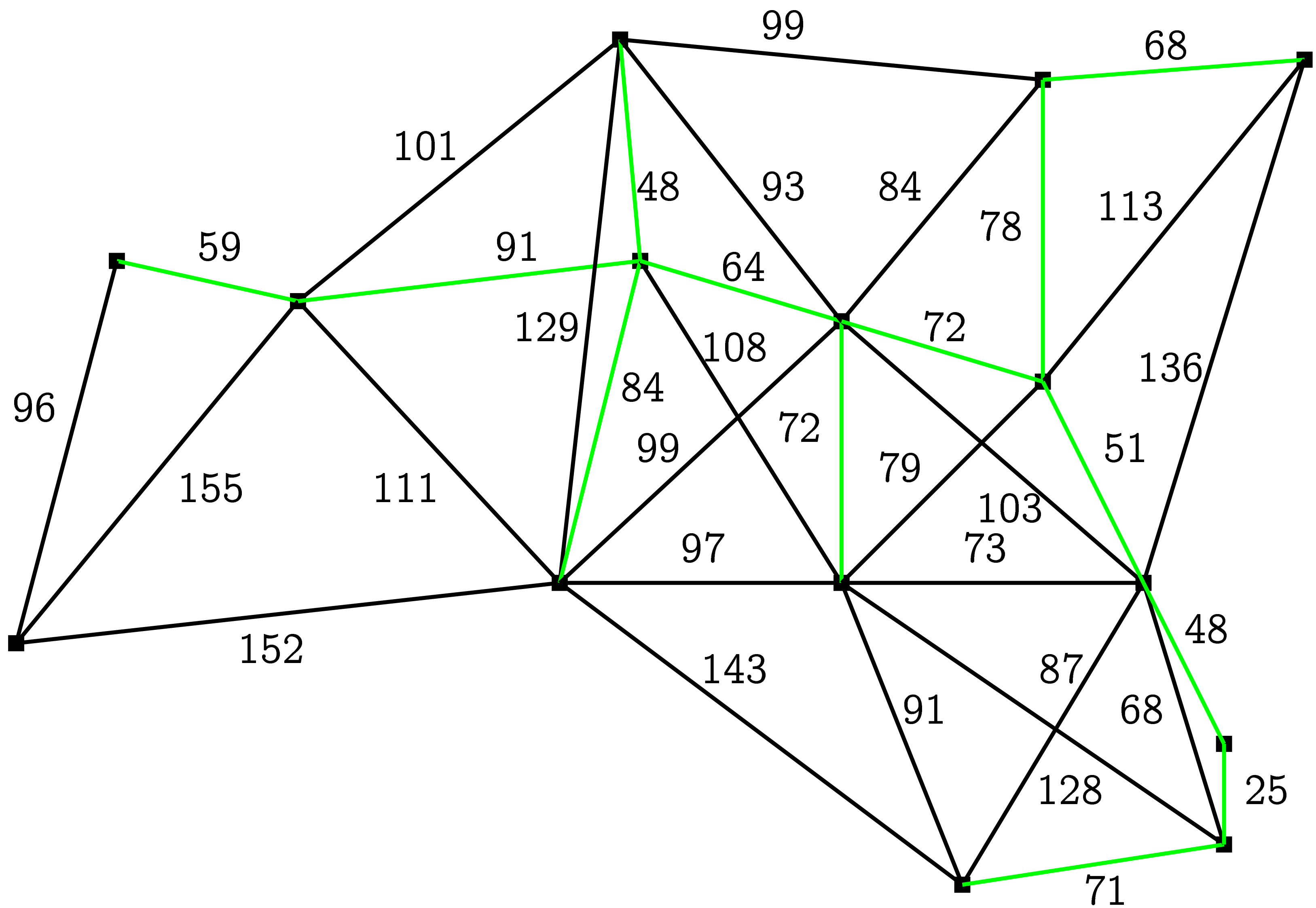
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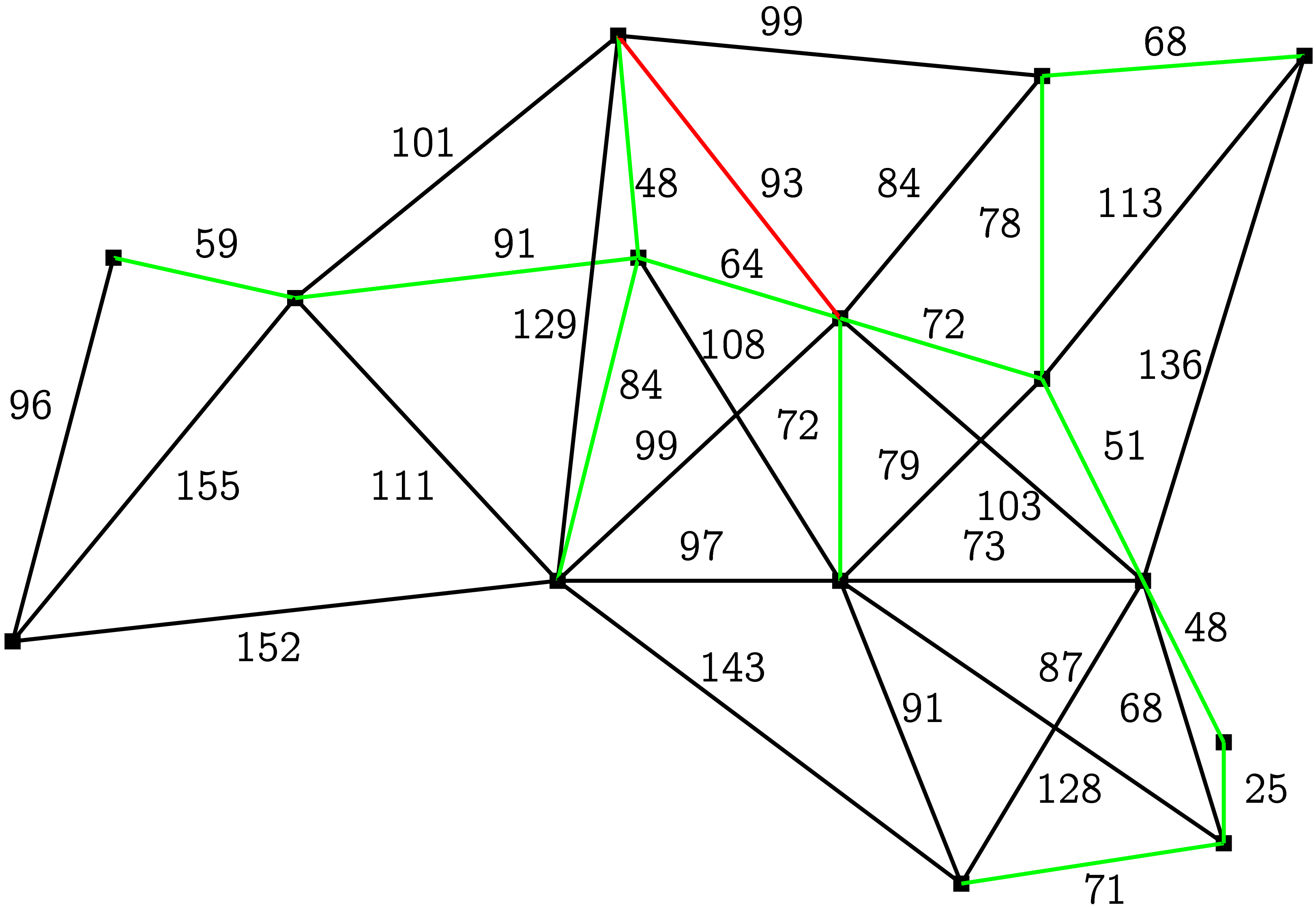
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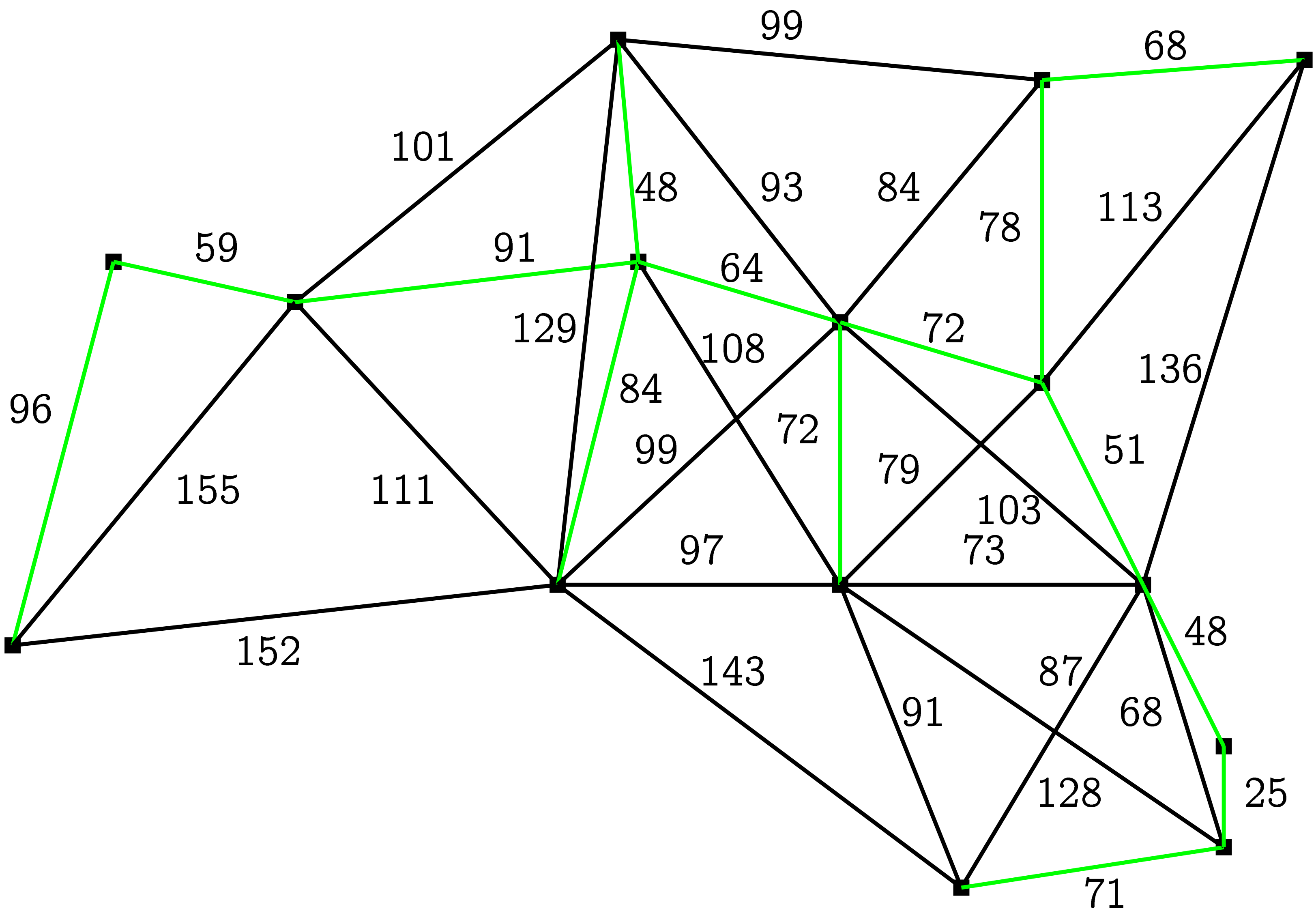
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Theorem. The presented algorithm is correct.

Proof. T is a (spanning) tree — it has no cycles, but does have n vertices and $n-1$ edges.

Assume that $w(T)$ is not minimal possible. Let T' be some minimal spanning tree of G. Let T' be such that is has the maximal possible number of edges in common with T.

Let $k \in \{1, \dots, n-1\}$ be the least number such that $e_k \notin E(T')$.

Let $S = T' \cup \{e_k\}$. The graph S has a cycle C.

Since T and T' have no cycles, we must have $e_k \in C$ and there exists an edge $e \in E(T') \setminus E(T)$ such that $e \in C$.

The graph $T'' = S \setminus \{e\}$ is connected and has $n-1$ edges, i.e. it is a spanning tree.

Edge e

- is different from $e_1, \ldots, e_{k-1}$,
- does not form a cycle together with $e_1, \ldots, e_{k-1}$ (since $e_1, \ldots, e_{k-1} \in E(T')$).

The edge e_k has minimal weight among the edges such that

- are different from $e_1, \ldots, e_{k-1}$,
- do not form a cycle together with $e_1, \ldots, e_{k-1}$.

Thus $w(e_k) \leq w(e)$.

We obtain $w(T'') = w(T') - w(e) + w(e_k) \leq w(T')$, i.e. T'' is a minimal weight spanning tree.

The tree T'' has more edges in common with T than T' does. A contradiction with the choice of T'. $\square$