Matchings and coverings

Consider a set X. We want to pair its elements.

The set of potential pairs is constrained by the relation

$$P \subseteq \{\{x, y\} \mid x, y \in X, x \neq y\},$$

showing which elements can be paired.

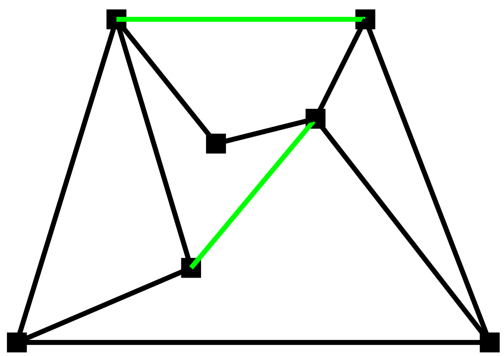
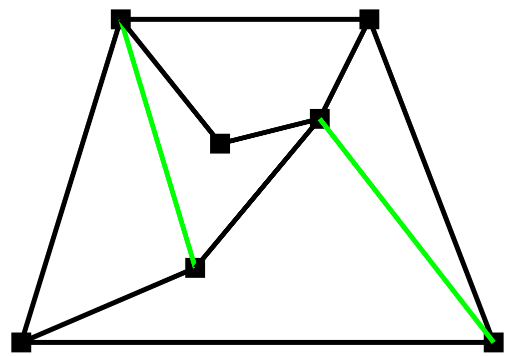
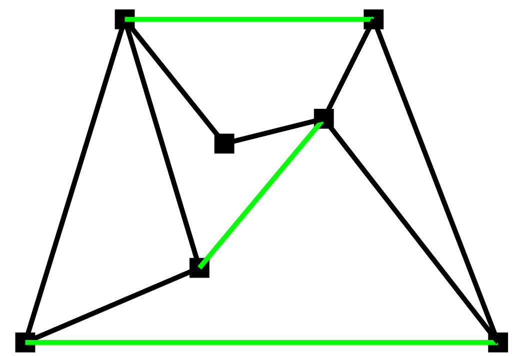
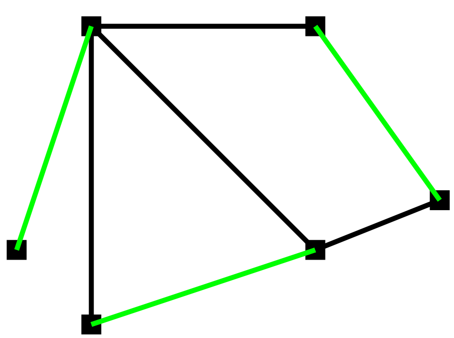
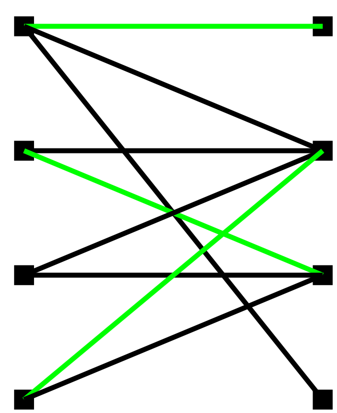
In this lecture, we assume (X, P) to be a simple graph.

Often (X, P) is a bipartite graph. E.g., X can be the set of lecture halls and potential times of particular lectures. P can indicate which halls can accommodate which lectures.

Let $G = (V, E)$ be a simple graph. *Matching (kooskõla)* in the graph G is a set $M \subseteq E$ of edges such that for each $v \in V$ we have $\deg_M(v) \leq 1$.

The matching is *maximal* if its cardinality is the largest possible.

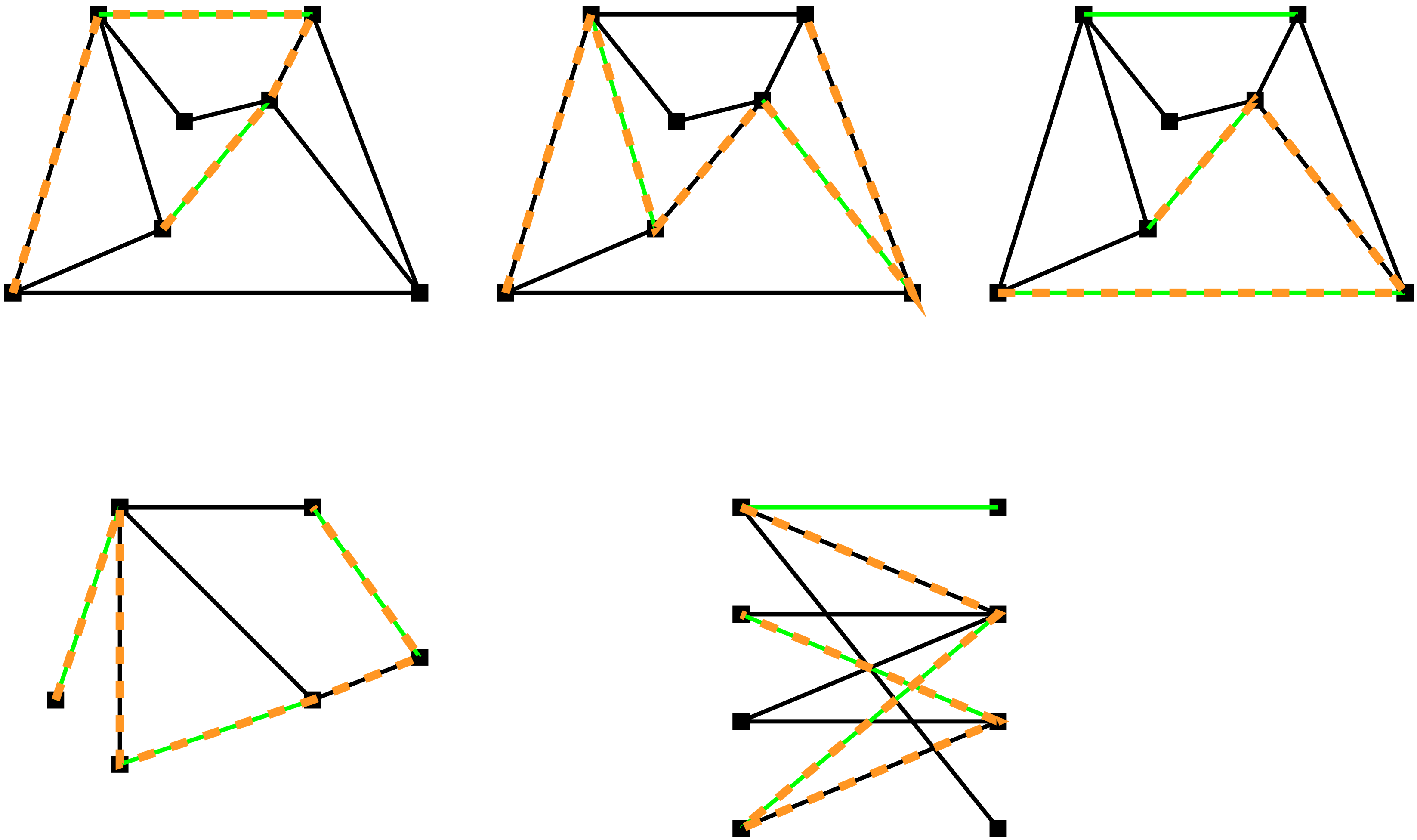
The matching M is *perfect (täielik)* if $\deg_M(v) = 1$ holds for every $v \in V$.

Let $G = (V, E)$ be a simple graph, $M \subseteq E$ a matching and P some path (with different endpoints) in the graph G.

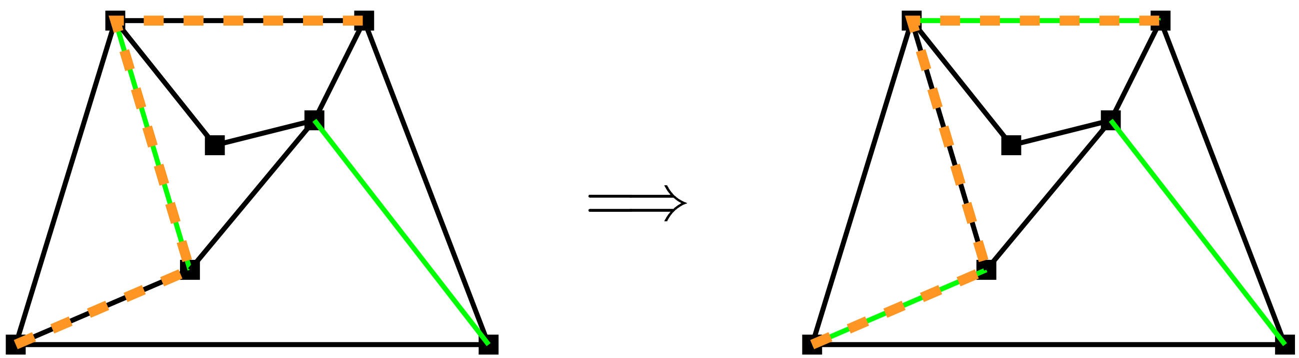
The path P is *M-alternating (vahelduv)* if its edges alternately belong to the sets M and $E \backslash M$.

The path P with endpoints x and y is *M-extensible (laienev)* if it is M-alternating and $\deg_M(x) = \deg_M(y) = 0$.



Theorem (Berge). Matching M in the graph $G = (V, E)$ is maximal iff there are no M-extensible paths in G.

Proof $\Rightarrow$. Assume to the contrary that there exists an M-extensible path P in G.



Consider P as a set of edges.

Let $M' = (M \backslash P) \cup (P \backslash M)$. Then $|M'| = |M| + 1$.

It is easy to verify that M' is a matching. Let $v \in V$, we will show that $\deg_{M'}(v) \leq 1$. There are three options.

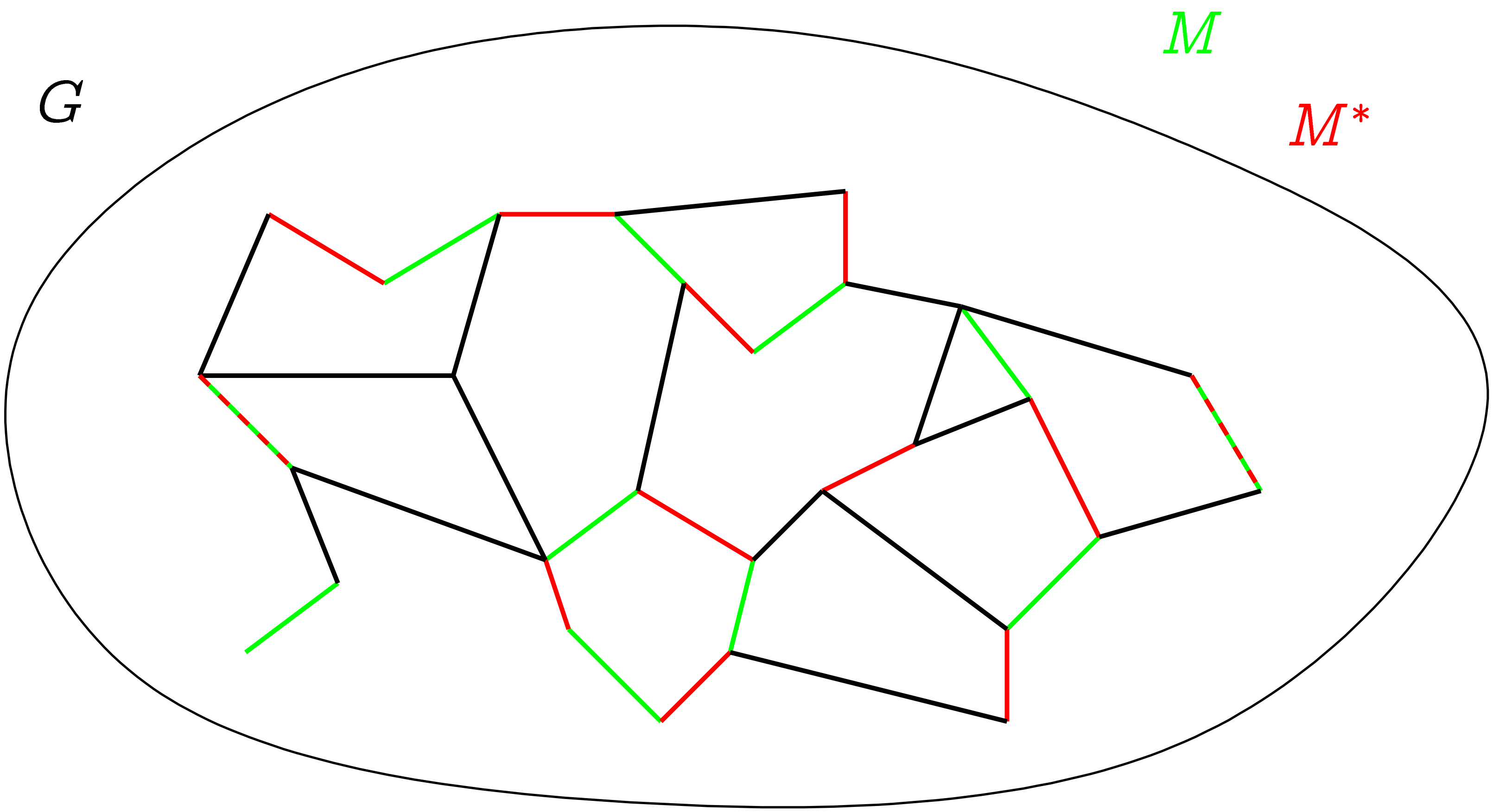
- v is not on the path P. Then $\deg_M(v) = \deg_{M'}(v)$. Indeed, let $e \in E$ be incident with v. As $e \notin P$, we have $e \in M \Leftrightarrow e \in M'$.

- v is an endvertex of P. Then $\deg_{M'}(v) = \deg_M(v)+1 = 1$.

- v is an internal vertex of P. Then $\deg_{M'}(v) = \deg_M(v) = 1$.

Proof $\Leftarrow$. We will construct an M-extensible path.

Let M^* be a maximal matching in G. Then $|M| < |M^*|$.
Consider the graph $H = (V, M \cup M^*)$.



For each $v \in V$ we have $\deg_H(v) \leq 2$. Possible connected components of H are:

- Isolated vertices.
- Paths.
  - Closed paths, i.e. cycles.
    - The edges of M and M^* alternately.
  - Open paths. Options:
    - A lonely edge $e \in M \cap M^*$.
    - The edges of M and M^* alternately. Options:
      - Having one end in M, another end in M^*.
      - Having both ends in M.
      - Having both ends in M^*.

Since $|M| < |M^*|$, there must exist a connected component of H having more edges from M^* than edges from M.

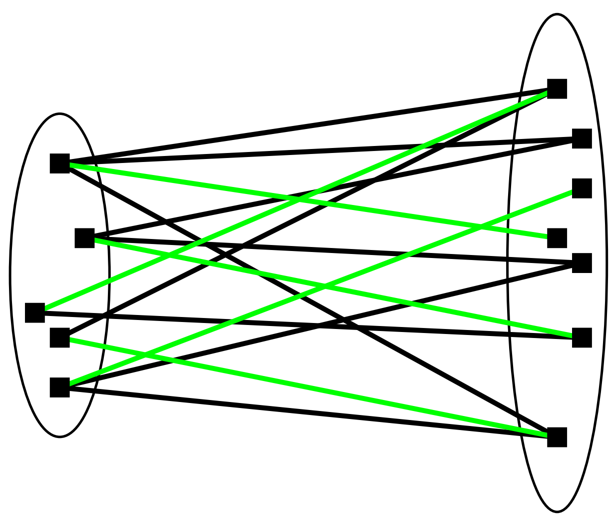
The only such components are open paths having both ends in M^*.

These paths are M-extensible. $\square$

Let $G = (V, E)$ be a graph and let $S \subseteq V$. *Neighbourhood (naabrus)* of S is the set

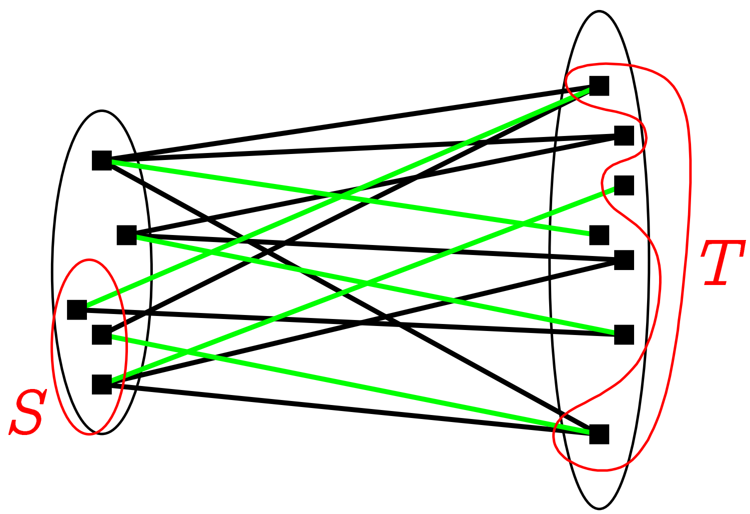
$$N(S) = \{w \mid w \in V, \exists e \in E, \exists v \in S : \mathcal{E}(e) = \{v, w\}\} \ .$$

Theorem (Hall). Let $G = (V, E)$ be a bipartite graph with vertex set partition to X and Y. The graph G has a matching M with the property $\forall x \in X : \deg_M(x) = 1$ iff for each $S \subseteq X$ the inequality $|N(S)| \geq |S|$ holds.



Proof $\Rightarrow$. Let M be a matching with the required property. Let $S \subseteq X$. Consider the set

$$T = \{y \mid y \in Y, \exists x \in S : (x, y) \in M\} \ .$$

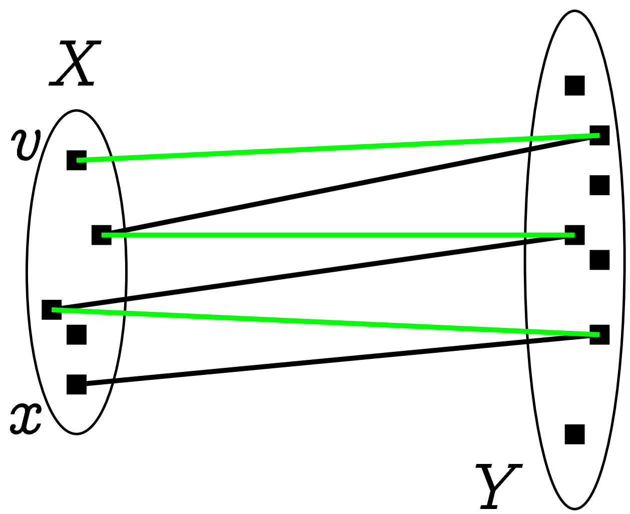


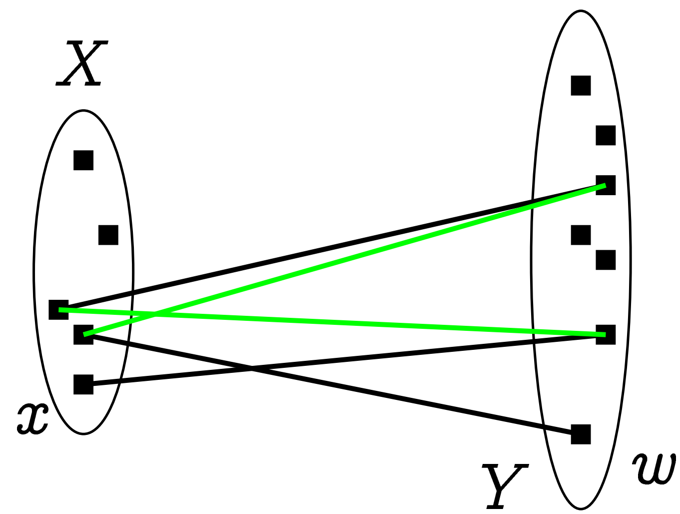
Then $|T| = |S|$, since each $x \in S$ defines a different y. We also have $T \subseteq N(S)$, consequently $|S| = |T| \leq |N(S)|$.

Proof $\Leftarrow$. Let M be some maximal matching. Assume to the contrary that there exists $x \in X$, such that $\deg_M(x) = 0$.

Let $S \subseteq X$ be the set of all vertices $v \in X$ such that there exists an M-alternating path from x to v. Note that $x \in S$.

Let $T \subseteq Y$ be the set of all vertices $w \in Y$ such that there exists an M-alternating path from x to w.





We will show that

I. $N(S) = T$;

II. $|S \backslash \{x\}| = |T|$.

As a consequence, we will get a contradiction:

$$|N(S)| = |T| = |S \backslash \{x\}| = |S| - 1 < |S| \quad .$$

Part I. Let $v \in S$ and let P be an M-alternating path from x to v. Note that the last edge on the path P belongs to M.

Let $w \in Y$ be a neighbour of vertex v. There are two options:

1. w is on the path P. The part of P from x to w is an M-alternating path from x to w. Thus $w \in T$.

2. w is not on the path P. Two options again:
   - $(v, w) \in M$. Then (v, w) is the last edge on the path P, because there are no other edges in M incident with v. Thus we are back to the 1st option.
   - $(v, w) \notin M$. Then P together with the edge (v, w) is an M-alternating path from x to w. Thus $w \in T$.

Part II. We will construct a bijection between $S \setminus \{x\}$ and T.

Let $v \in S \setminus \{x\}$. Then there is an edge $e \in M$ incident with v (the last edge on the M-alternating path from x to v). We let the other endvertex w of e to correspond to v. We proved on the last slide that $w \in T$.

Let $w \in T$. If there was no edge $e \in M$ being incident with w, we would get an M-extensible path from x to w. Berge Theorem forbids this, thus we have such an edge e.

We let the other endvertex v of e to correspond to w. Obviously, $v \in S$. Also, $v \neq x$, since the other endvertex of e is not x, because $\deg_M(x) = 0$. $\square$

Corollary. Regular (i.e. with all vertex gedrees equal) bipartite non-null graph has a perfect matching.

Proof. Let $G = (V, E)$ be a bipartite graph with partition X and Y. Let $k > 0$ be the degree of all the vertices. Since

$$|X| \cdot k = \sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y) = |Y| \cdot k,$$

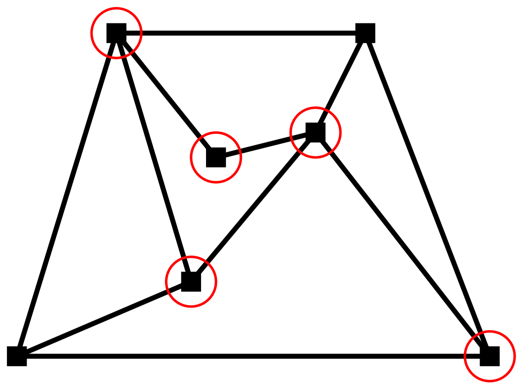
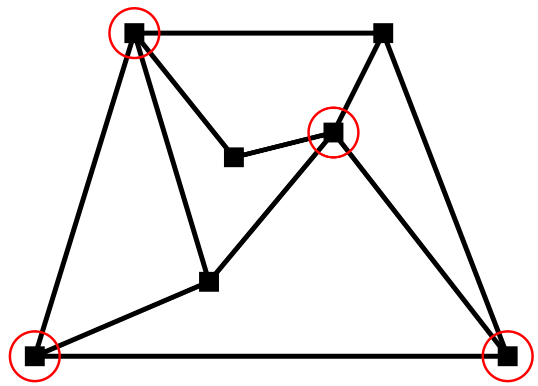
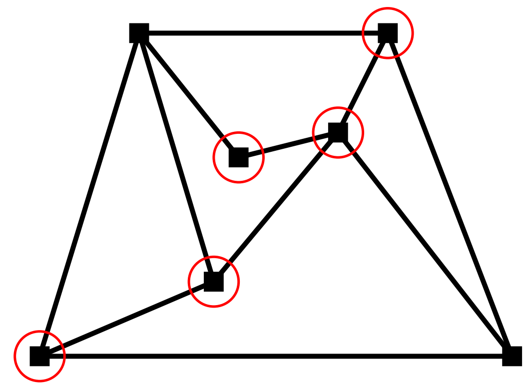
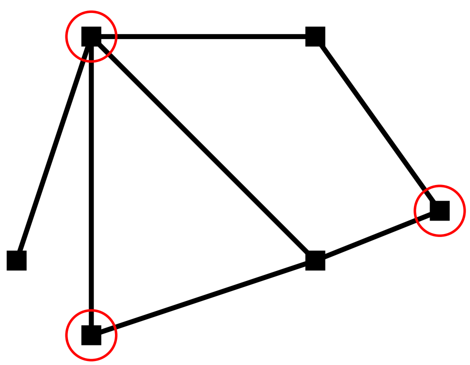
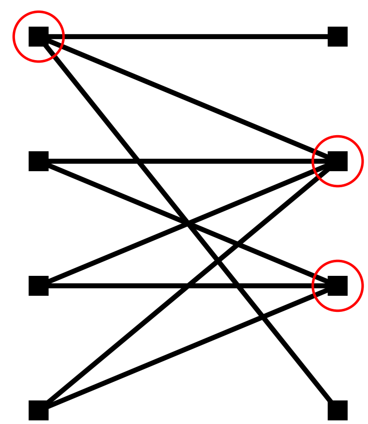
we have $|X| = |Y|$. Let $S \subseteq X$. Since

$$|S| \cdot k = \sum_{x \in S} \deg(x) \leq \sum_{y \in N(S)} \deg(y) = |N(S)| \cdot k,$$

we get $|S| \leq |N(S)|$. Thus there exists a matching M such that $\deg_M(x) = 1$ for each $x \in X$. Since $|X| = |Y|$, we also have $\deg_M(y) = 1$ for each $y \in Y$. $\square$

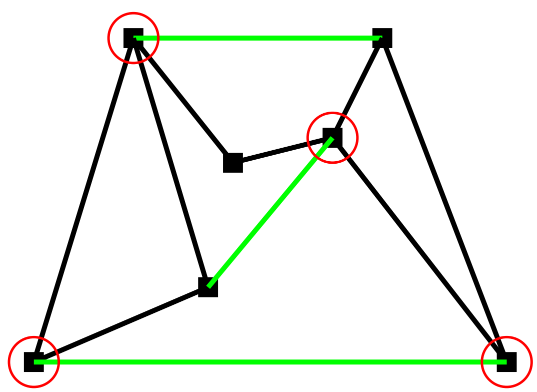
Let $G = (V, E)$ be a simple graph. *Cover (kate)* in graph G is the set $K \subseteq V$ of vertices such that each $e \in E$ is incident with some vertex from K.

Cover is *minimal* if its cardinality is the smallest possible.

Proposition. Let $G = (V, E)$ be a simple graph, M some of its matchings and and K some of its covers. Then $|M| \leq |K|$.

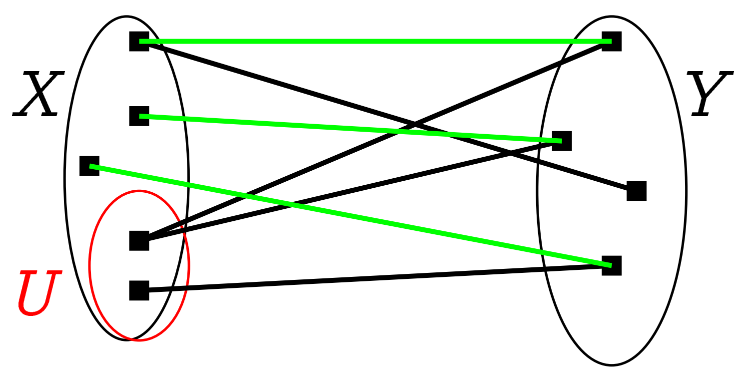
Proof. For each edge $e \in M$, there exists a vertex $v \in K$ such that e is incident with v. For different edges these vertices differ, since the edges of M can not have common endvertices. $\square$



Theorem (König). Let $G = (V, E)$ be a bipartite graph. Then the cardinalities of maximal matchings and minimal covers are equal.

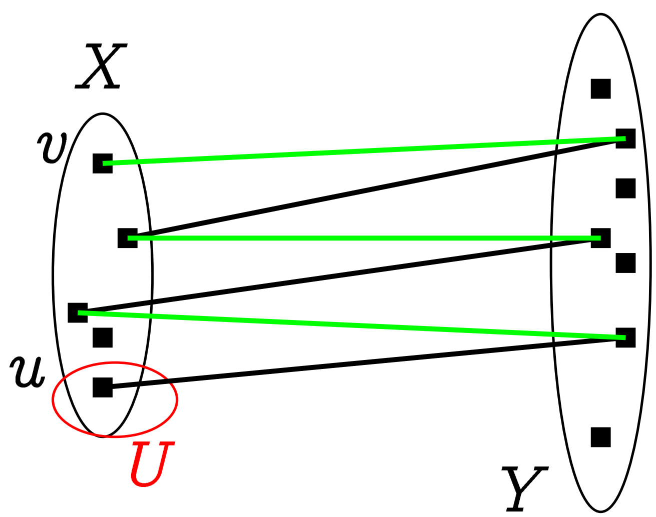
Proof. Let X and Y be the partition of G and let M be one of its maximal matchings. We will construct a cover K such that $|M| = |K|$.

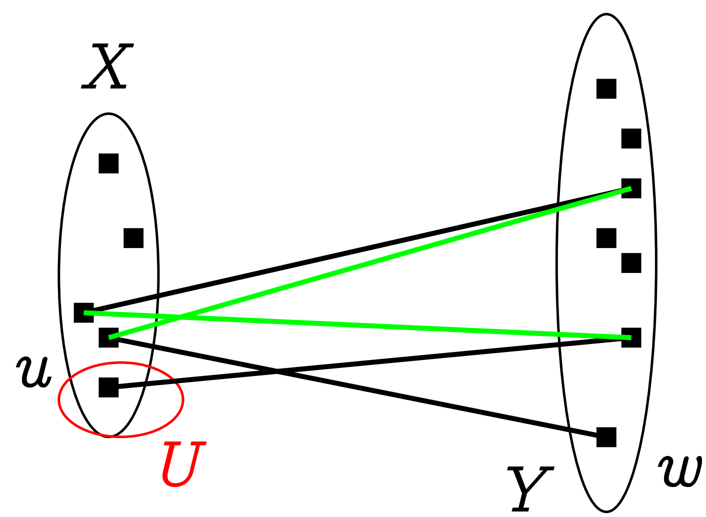
Let $U \subseteq X$ be the set of such vertices $u \in X$ that $\deg_M(u) = 0$. Then $|M| = |X \setminus U|$.



Let $S \subseteq X$ be the set of such vertices $v \in X$ that for some $u \in U$ there exists an M-alternating path from u to v. Then $U \subseteq S$.

Let $T \subseteq Y$ be the set of all such vertices $w \in Y$ that for some $u \in U$ there exists an M-alternating path from u to w.

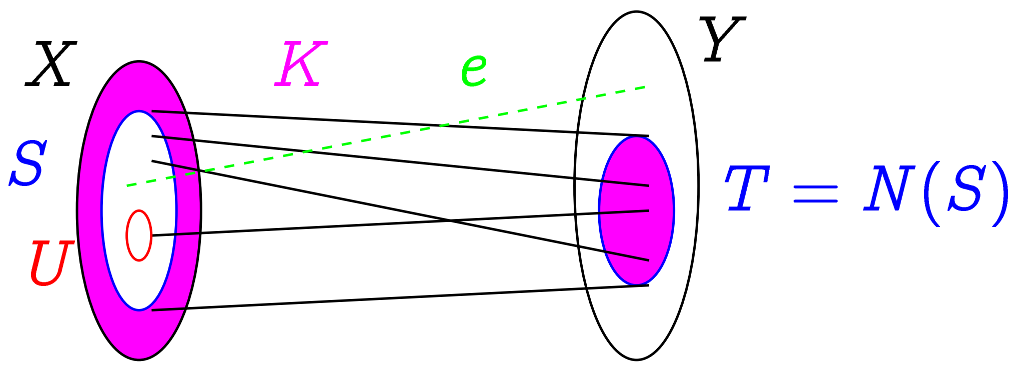




Similarly to the proof of Hall's Theorem we can prove $N(S) = T$ and $|T| = |S \backslash U|$.

Let $K = T \cup (X \backslash S)$. Then K is a cover.

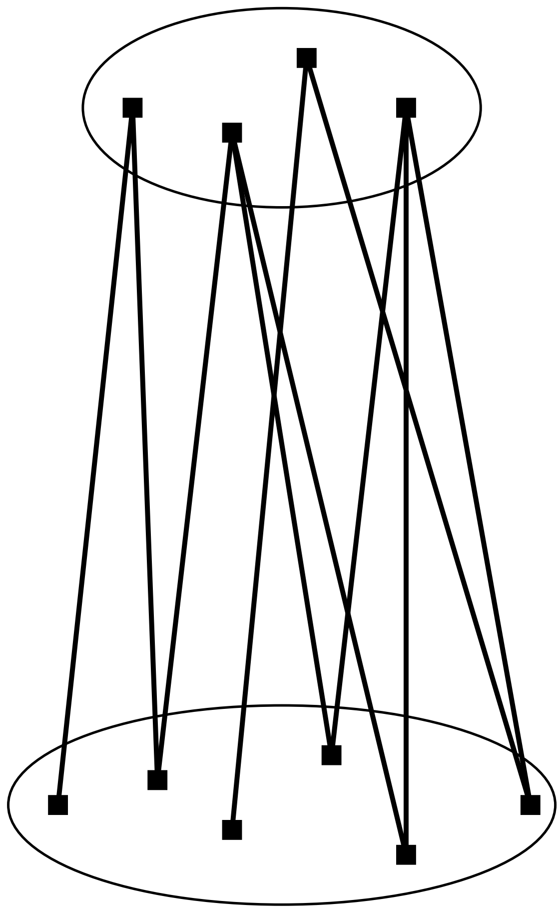
Indeed, assume that there is an edge $e \in E$ that is not incident with any vertex of K. Then one endvertex of e is in S and another one in $Y \backslash T$. Contradiction with the observation $N(S) = T$.



$$|K| = |T| + |X \backslash S| = |S \backslash U| + |X \backslash S| = |X \backslash U| = |M| \ .$$

□

How to find maximal matchings in bipratite graphs?



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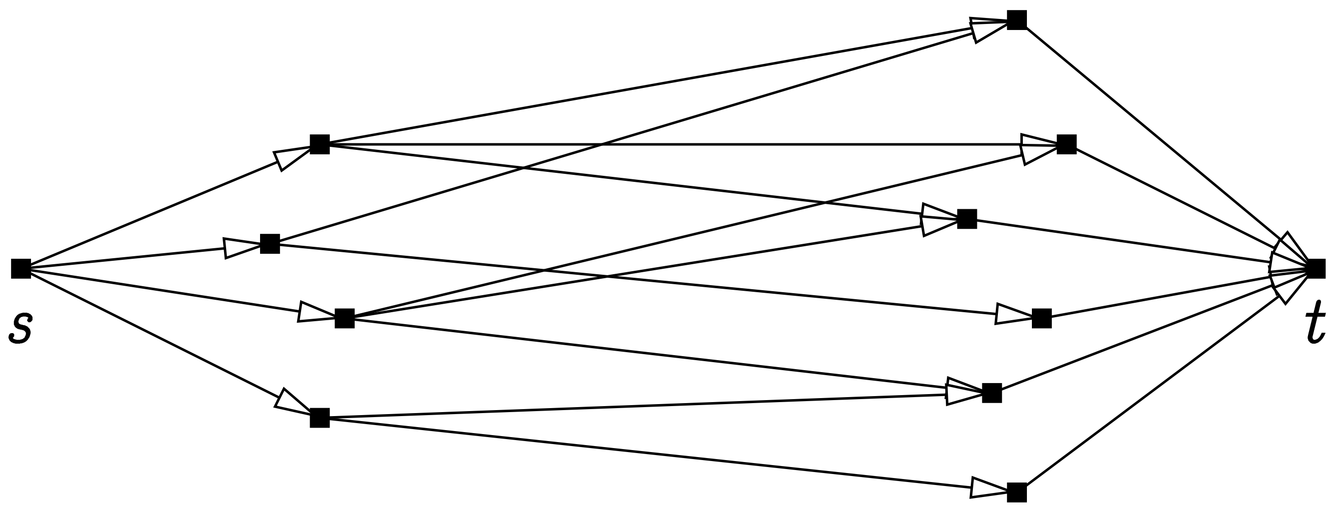

kõigi kaarte läbilaskevõime on 1

Ford-Fulkerson algorithm gives a maximal flow, assigning each arc an integer.

Maximal matching in the original graph is determined by these edges that were assigned 1.