Solutions for the reattempt of the 3rd test in Graphs

January 15th, 2009

Exercise 1. Does there exist an $n \in \mathbb{N}$, such that there exist simple planar bipartite graphs G_1 and G_2 with n vertices and $2n - 4$ edges, such that $G_1 \not\cong G_2$?

Answer: yes. Both $K_{2,4}$ and $K_{3,3} - e$ (the graph $K_{3,3}$ where one of the edges has been deleted) are planar and contain 6 vertices and $8 = 2 \cdot 6 - 4$ edges.

Exercise 2. For any odd $n \in \mathbb{N}$, define the simple graphs G_n as follows:

- the set of vertices V_n of G_n is $\{1, 3, 5, \ldots, n\}$;
- two numbers $x, y \in V_n$ are connected with an edge iff $\gcd(x, y) > 1$.

For which values of n is G_n planar?

Answer: G_n is planar iff $n \leq 25$. A planar drawing of G_{25} is depicted in Fig. 1. But the graph G_{27} contains a clique of size five; it is formed by the vertices $3, 9, 15, 21, 27$.

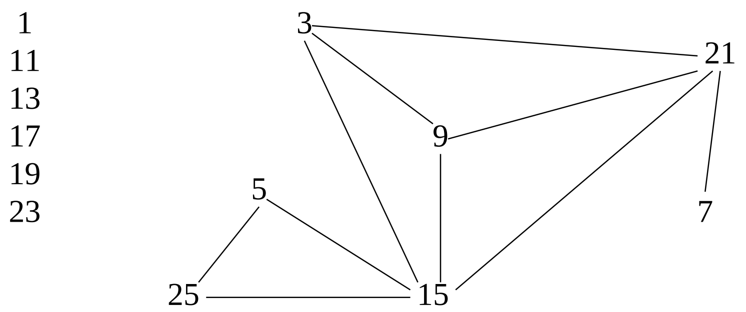

Figure 1:

Exercise 3. The Ramsey number $r(k, l)$ is defined as the smallest n, such that for any coloring of the edges of K_n with two colors, there exists a monochromatic copy of K_k of the first color, or a monochromatic copy of K_l of the second color. We can generalize $r(k, l)$ as follows: for any two (simple) graphs G_1, G_2 let $r(G_1, G_2)$ be the smallest n, such that for any coloring of the edges of K_n with two colors, there is an $i \in \{1, 2\}$, such that G_i is a subgraph (not necessarily induced) of the graph made up of the edges of i-th color.

Show that if T is any tree with n vertices, then
$r(K_m, T) \geq (m - 1)(n - 1) + 1$.

Proof. To show that $r(K_m, T)$ is at least $(m-1)(n-1)+1$ we have to show how to color the edges of $K_{(m-1)(n-1)}$ with two colors, such that it contains neither K_m of the first color nor T of the second color.

We partition the vertices of $K_{(m-1)(n-1)}$ into $(m-1)$ parts, each containing $(n-1)$ vertices. We color an edge (u,v) with the first color iff u and v belong to different parts. We color an edge (u,v) with the second color iff u and v belong to the same part.

The edges of the first color form the complete $(m-1)$-partite graph $K_{n-1,n-1,\ldots,n-1}$. It contains no subgraph isomorphic to K_m. Indeed, for any m vertices, at least two of them must belong to the same part and thus the edge connecting them is not colored with the first color.

The edges of the second color form $(m-1)$ independent copies of the graph K_{n-1}. As no connected component of this graph contains n or more vertices, it cannot contain the n-vertex connected graph T as a subgraph.

Exercise 4. Find the chromatic polynomial of the $2n$-vertex graph G_n, depicted in Fig. 2. below.

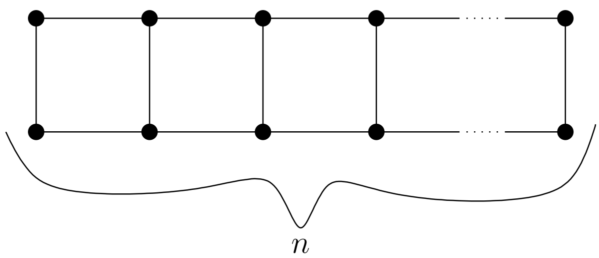

Figure 2:

Answer: $k(k-1)(k^2-3k+3)^{n-1}$.

Lemma. Let G be a graph, let its chromatic polynomial $P_G(k)$ be $p(k)$. Let the graph G' be constructed from G by

- adding to it two new vertices u and v and the edge e between them;
- adding to it the edges (a,u) and (b,v), where a and b are two different vertices of G, such that the edge (a,b) is contained in G. (See Fig. 3)

Then $P_{G'}(k) = (k^2-3k+3)\cdot p(k)$.

Proof of the lemma. We know that $P_{G'}(k) = P_{G'-e}(k) - P_{G'/e}(k)$. The chromatic polynomial $P_{G'-e}(k)$ is $(k-1)^2 \cdot p(k)$. Indeed, to color the vertices of $G'-e$, we first color the graph G, there are $p(k)$ possibilites for that. Afterwards, we have $(k-1)$ possible colors for the vertex u and $(k-1)$ possible colors for the vertex v. The chromatic polynomial $P_{G'/e}(k)$ is $(k-2)\cdot p(k)$.

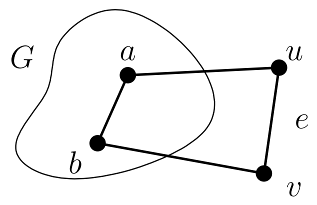

Figure 3:

Indeed, to color the vertices of G'/e we again start by coloring the vertices of G. After that we have one more vertex to color — the one obtained by contracting the edge e. This vertex has two neighbours — the vertices a and b. As the vertices a and b have different colors in a valid coloring of G, there are $(k-2)$ possible colors left over for the new vertex. Finally, $(k-1)^2-(k-2)=k^2-3k+3$.

To prove the answer of the exercise, we note that the graph G_n has been obtained by applying the construction of the lemma $(n-1)$ times to the graph K_2. The chromatic polynomial of K_2 is $k(k-1)$; this has to be multiplied $(n-1)$ times by k^2-3k+3.