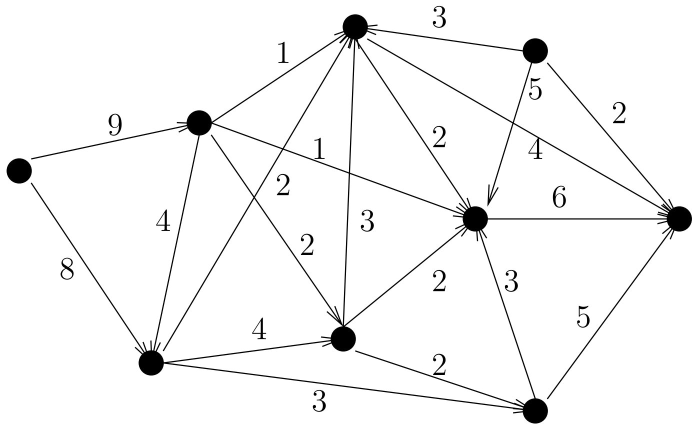
Solutions for the reattempt of the 2nd test in Graphs

January 19th, 2009

Exercise 1. Find the maximum flow and the minimum cut in the following network:



Just use Ford-Fulkerson's algorithm…

Exercise 2. Let $\mathcal{X}$ be the set of all subsets of $\{1, \ldots, n\}$. Let $\mathcal{X}_k \subseteq \mathcal{X}$ contain all subsets of $\{1, \ldots, n\}$ with exactly k elements.

Let $k \leq n/2$. The sets $\mathcal{X}_k$ and $\mathcal{X}_{n-k}$ have the same number of elements. Indeed, $\binom{n}{k} = \binom{n}{n-k}$. Does there exist a bijection from $\mathcal{X}_k$ to $\mathcal{X}_{n-k}$, such that each $X \in \mathcal{X}_k$ is mapped to one of its supersets?

Answer: yes. Consider the following bipartite graph. Let $\mathcal{X}_k$ and $\mathcal{X}_{n-k}$ be the two parts and let there be an edge between some $X \in \mathcal{X}_k$ and $Y \in \mathcal{X}_{n-k}$ iff $X \subseteq Y$.

By symmetry, all vertices in $\mathcal{X}_k$ have the same degree d_k. Also by symmetry, all vertices in $\mathcal{X}_{n-k}$ have the same degree d_{n-k}. The total number of edges in the graph is $|\mathcal{X}_k| \cdot d_k$ which is also equal to $|\mathcal{X}_{n-k}| \cdot d_{n-k}$. As $|\mathcal{X}_k| = |\mathcal{X}_{n-k}|$, we also have $d_k = d_{n-k}$.

Hence we have constructed a regular bipartite graph. By one of the corollaries of Hall's theorem, it has a perfect matching M. This matching M associates a $(n-k)$-element set Y to each k-element set X, such that X and Y are connected by an edge, i.e. $X \subseteq Y$. The edges of M define the bijection we're looking for.

Exercise 3. Show that a tree cannot have two different perfect matchings.

Proof. Let M and M^* be two perfect matchings of the tree T. I.e. $\deg_M(v) = \deg_{M^*}(v) = 1$ for any vertex $v \in V(T)$. Consider the symmetric difference $M \triangle M^*$. For some $v \in V(T)$, the value of $\deg_{M \triangle M^*}(v)$ can be one of the following:

- 0, if the edge incident to v in M is the same as the edge incident to v in M^*;
- 2, if the edge incident to v in M is different from the edge incident to v in M^*.

Consider the subgraph T' of T containing all vertices, but only edges in $M \triangle M^*$. As the vertex degrees in T' are in $\{0, 2\}$, the connected components of T' are either isolated vertices or cycles. But T' cannot contain cycles, otherwise its supergraph, the tree T would also contain cycles. Hence all connected components of T' are isolated vertices, thus T' has no edges, thus $M \triangle M^* = \emptyset$, thus $M = M^*$.

Exercise 4. We say that a graph is *uniquely* k-edge colorable, if its edges can be colored with k colors in exactly one way (*modulo* renaming of colors). In other words, all colorings with k colors give the same partition of edges into matchings.

Show that uniquely 3-edge colorable 3-regular graphs are Hamiltonian.

Proof. Let $G = (V, E)$ be a uniquely 3-edge colorable 3-regular graph. Let $E = E_1 \dot\cup E_2 \dot\cup E_3$, where $\{E_1, E_2, E_3\}$ is the partition of E induced by any 3-coloring of the edges of G. Note that each one of E_1, E_2 and E_3 is a perfect matching, because each vertex must have an edge of each color incident to it (the number of colors equals the degree of all vertices).

Consider the graph $G' = (V, E_1 \cup E_2)$. In this graph, the degree of all vertices is 2 (because $\deg_{E_1}(v) = \deg_{E_2}(v) = 1$ and $E_1 \cap E_2 = \emptyset$). Hence the connected components of G' are cycles. Let $C_1, C_2, \ldots \subseteq E_1 \cup E_2$ be the cycles of G'. We can consider a new coloring of G, by swapping the colors 1 and 2 in just the cycle C_1. This defines us a new partitioning of edges:

- $E_1' = (E_1 \backslash C_1) \cup (C_1 \cap E_2)$;
- $E_2' = (E_2 \backslash C_1) \cup (C_1 \cap E_1)$;
- $E_3' = E_3$.

If C_1 contains all edges of $E_1 \cup E_2$, then this is the same partition as before (we have $E_1' = E_2$ and $E_2' = E_1$). Otherwise we get a different partition. But according to our premises, no other partitions are possible. Hence the graph G' is made of a single cycle C_1. This cycle passes all vertices of G' (and G) i.e. it is a Hamiltonian cycle.