Solutions for the 3rd test in Graphs

December 19th, 2008

Exercise 1. Do simple planar graphs with n vertices and (exactly) $3n-6$ edges exist for each $n \geq 3$?

Answer: yes. Proof: induction over n.

Base. $n = 3$. We need to present a planar simple graph with 3 vertices and $3 \cdot 3 - 3 = 3$ edges. The example is K_3.

Step. Suppose that for some n, there exists a simple planar graph G with n vertices and $3n-6$ edges. Let F be a face in a planar drawing of G. As G is simple, F has at least three sides. Define a new graph G' by adding a new vertex to G, and connecting it with three different vertices at F. The graph G' remains simple. Also, G' is planar — to draw it, first draw G and then add the new vertex to the middle of F and connect it with the vertices at the sides (see Fig. 1). The graph G' has $n+1$ vertices and $3n-6+3 = 3(n+1)-6$ edges.

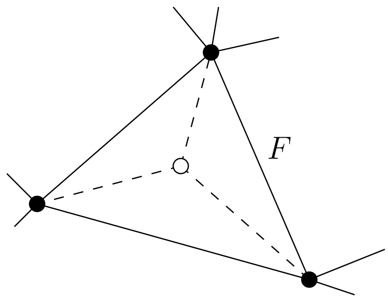

Figure 1:

Exercise 2. For any $n \in \mathbb{N}$, define the simple graphs G_n as follows:

- the set of vertices of G_n is $\{1, 2, \ldots, n\}$;
- two numbers $x, y \in \{1, \ldots, n\}$ are connected with an edge iff $x \mid y$ or $y \mid x$.

For which values of n is G_n planar?

Hint. A graph is called *outerplanar* if it can be drawn on a plane so, that there is a face containing all vertices of the graph. A graph is outerplanar iff it contains no subgraphs homeomorphic to K_4 or $K_{2,3}$.

Answer: G_n is planar iff $n \leq 14$.

Notice first that in G_n, the vertex 1 is connected to all other vertices. Suppose that G_n is planar. If we delete the vertex 1 from the drawing of G_n, we end up with a face that contains all other vertices (see Fig. 2). Therefore, if G_n is planar, then $G_n \backslash 1$ is outerplanar. The opposite also holds — G_n can be planar only if $G_n \backslash 1$ is outerplanar.

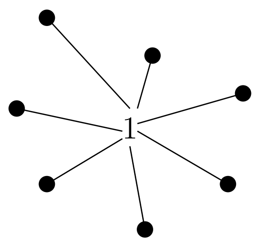

Figure 2:

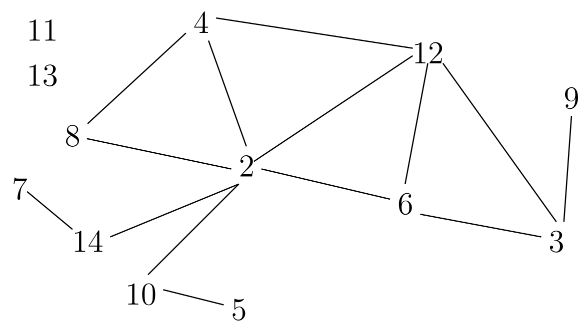

Figure 3:

Fig. 3 shows that $G_{14} \backslash 1$ is outerplanar. But if we add the vertex 15 to this graph, giving us $G_{15} \backslash 1$, then a homeomorphic copy of K_4 appears (see Fig. 4). Its vertices are 2, 3, 6 and 12. Five of the six edges of K_4 are already present in $G_{15} \backslash 1$. The edge between 2 and 3 is represented by the path 2—10—5—15—3.

Exercise 3. Show that if $k \geq 2$, then $r(k, l+1) - r(k, l) \geq 1$ for any $l \geq 1$.

Proof. Let G be any graph with $r(k, l+1) - 1$ vertices. We show that either $K_k \hookrightarrow G$ or $O_l \hookrightarrow G$. In this way, we will have shown that $r(k, l) \leq r(k, l+1) - 1$.

Let G' be a graph that is obtained from G by adding to it an isolated vertex v. The graph G' has $r(k, l+1)$ vertices, hence at least one of the following must hold:

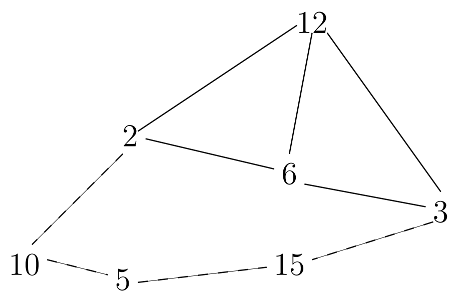

Figure 4:

- $K_k \hookrightarrow G'$. But this induced copy of K_k cannot include v, because the degree of v is less than $k-1 \geq 1$. Hence the same induced copy is also present in G.

- $O_{l+1} \hookrightarrow G'$. If this induced copy does not include the new vertex v then it is also present in G. And even if this induced copy of O_{l+1} contains v, it still gives us an induced subgraph O_l in $G = G' \backslash v$.

Exercise 4. How many valid colorings with three colors does graph in Fig. 5 have?

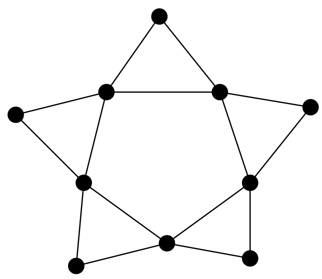

Figure 5:

Answer: 30. Indeed, if we have three colors, then this graph has the same number of colorings as the cycle C_5 because the colors of the "inner" five vertices uniquely determine the colors of the "outer" five. Indeed, if we have picked the colors for inner five vertices, then each of the outer vertices has two neighbours that have been colored with different colors. The third color is then the only one that has remained available for the outer vertex.

By exercise 120 of the course book, $P_{C_5}(k) = (k-1)^5 - (k-1)$. If $k = 3$ then this expression is equal to 30.