Graafid, 1. kontrolltöö

23. märts 2011

Ülesanne 1. Graafi $G = (V, E)$ tippu $v \in V$ nimetame selle graafi *tsentriks*, kui suurus $\max_{u \in V} d(u, v)$ on selle tipu jaoks minimaalne võimalik (graafi G tippude hulgas). Tippu $w \in V$ nimetame selle graafi *barütsentriks*, kui suurus $\sum_{u \in V} d(u, w)$ on selle tipu jaoks minimaalne võimalik (graafi G tippude hulgas).

Konstrueeri graaf G ning leia selles graafis tipud v ja w nii, et

- v on graafi G tsenter;
- w on graafi G barütsenter;
- $d(v, w) \geq 5$.

Ülesanne 2. Olgu $G = (V, E)$ lihtgraaf ja $U \subseteq V$. Defineerime operatsiooni G^U järgmiselt: $G^U = (V, E')$ on lihtgraaf, kus servade hulk E' on antud järgmiste reeglitega:

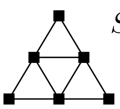
- Kui tipud $u, v \in V$ kuuluvad mõlemad hulka U või hulka $V \backslash U$, siis on graafis G^U tippude u ja v vahel serv parajasti siis, kui graafis G on nende tippude vahel serv.
- Kui tippudest u ja v kuulub üks hulka U ja teine hulka $V \backslash U$, siis on graafis G^U tippude u ja v vahel serv parajasti siis, kui graafis G nende tippude vahel serva ei ole.

Olgu $G = (V, E)$ paaritu arvu tippudega lihtgraaf. Näita, et leidub tipuhulk $U \subseteq V$, nii et G^U on Euleri graaf.

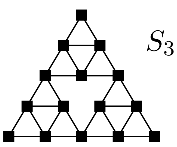
Ülesanne 3. Defineerime lihtgraafid $S_1, S_2, S_3, \ldots$, mida võiks nimetada *Sierpinski graafideks*. Graafiks S_1 on K_3. Nimetame selle graafi kolme tippu *vasakuks*, *paremaks* ja *ülemiseks*. Graafi S_{i+1} saame kui graafi S_i kolme eksemplari (mida nimetame samuti *vasakuks*, *paremaks* ja *ülemiseks*) ühendi, kus järgmised tipupaarid loeme üheksainsaks tipuks:

- S_i vasaku eksemplari ülemine tipp ja S_i ülemise eksemplari vasak tipp;
- S_i parema eksemplari ülemine tipp ja S_i ülemise eksemplari parem tipp;
- S_i parema eksemplari vasak tipp ja S_i vasaku eksemplari parem tipp.

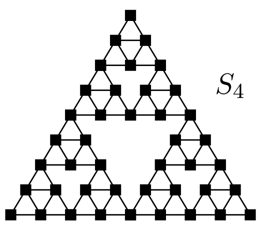
Graafi S_{i+1} tipp X (kus $X \in \{\text{vasak}, \text{parem}, \text{ülemine}\}$) on S_i X eksemplari X tipp. All on toodud mõned näited graafidest S_i. Näita, et kõik graafid S_i on Hamiltoni.



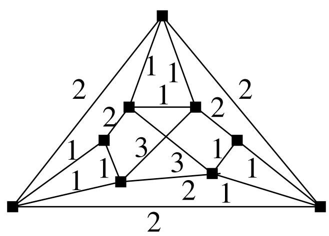
S_2



S_3



S_4



Ülesanne 4. Mitu erinevat minimaalse kaaluga aluspuud leidub ülalkujutatud graafil?

(Paberkandjal) materjale tohib kasutada.

Kõik ülesanded on võrdse kaaluga.

Graphs, 1st test
23 March 2011

Exercise 1. The *centre* of a graph $G = (V, E)$ is its vertex $v \in V$ that achieves the minimal possible value (among that graph's vertices) for the quantity $\max_{u \in V} d(u, v)$. The *barycentre* of a graph $G = (V, E)$ is its vertex $w \in V$ that achieves the minimal possible value for the quantity $\sum_{u \in V} d(u, w)$. Construct a graph G and find its vertices v and w so, that

- v is a centre of the graph G;
- w is a barycentre of the graph G;
- $d(v, w) \geq 5$.

Exercise 2. Let $G = (V, E)$ be a simple graph and $U \subseteq V$. Define the operation G^U as follows: $G^U = (V, E')$ is a simple graph where the set E' is given by the following rules:

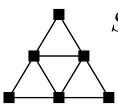
- If two vertices $u, v \in V$ both belong to U or both belong to $V \backslash U$, then they are adjacent in G^U iff they are adjacent in G.
- If one of the vertices $u, v \in V$ belongs to U and the other one belongs to $V \backslash U$, then they are adjacent in G^U iff they are not adjacent in G.

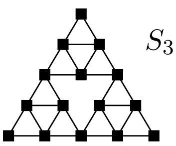
Let $G = (V, E)$ be a simple graph with an odd number of vertices. Show that there exists $U \subseteq V$, such that G^U is Eulerian.

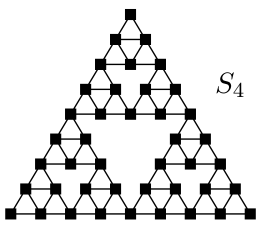
Exercise 3. Let us define simple graphs $S_1, S_2, S_3, \ldots$ that might be called *Sierpinski graphs*. Graph S_1 equals K_3. Let us call its three vertices the *left vertex*, the *right vertex* and the *top vertex*. The graph S_{i+1} is obtained as the union of three copies of S_i (that we also call *left*, *right* and *top*, respectively), where we identify the following pairs of vertices:

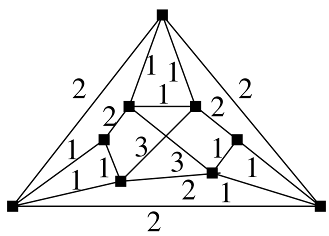
- the top vertex of the left S_i and the left vertex of the top S_i;
- the top vertex of the right S_i and the right vertex of the top S_i;
- the right vertex of the left S_i and the left vertex of the right S_i.

The X vertex (where $X \in \{\text{left}, \text{right}, \text{top}\}$) of S_{i+1} is the X vertex of the X copy of S_i. Some examples of the graphs S_i are given below. Show that all graphs S_i are Hamiltonian.


S_2


S_3


S_4



Exercise 4. How many different minimum-weight spanning trees does the graph above have?

This test is open-book.

All exercises are worth the same amount of points