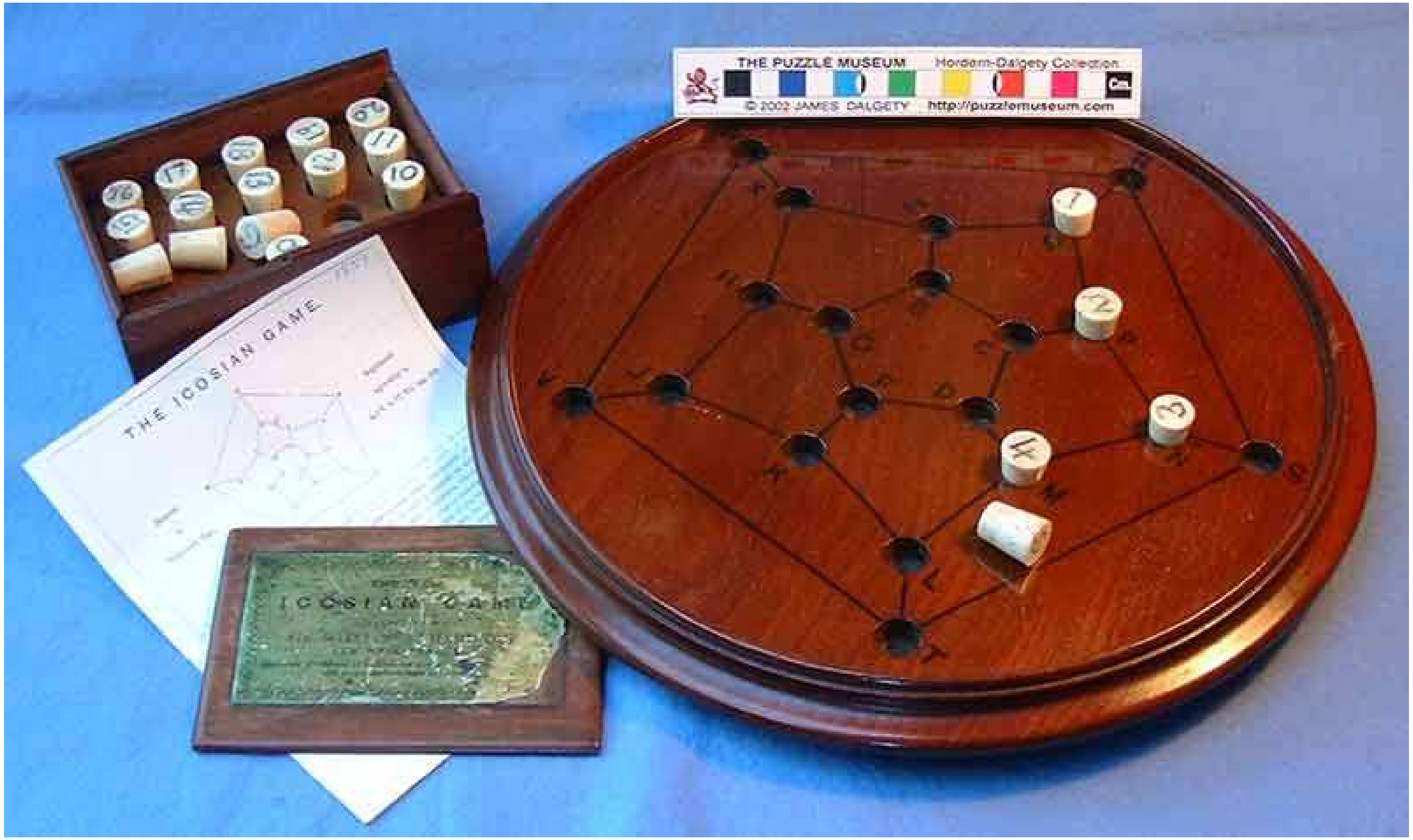
Hamiltonian graphs

Icosian game by sir William Rowan Hamilton, 1857



• *Hamiltonian cycle* in graph G is a cycle that passes through each vertex exactly once.

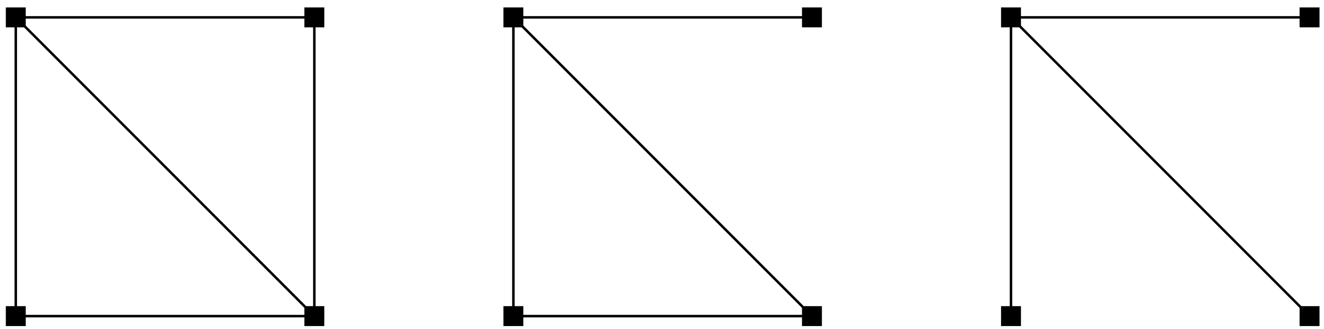
• *Hamiltonian walk* in graph G is a walk that passes through each vertex exactly once.

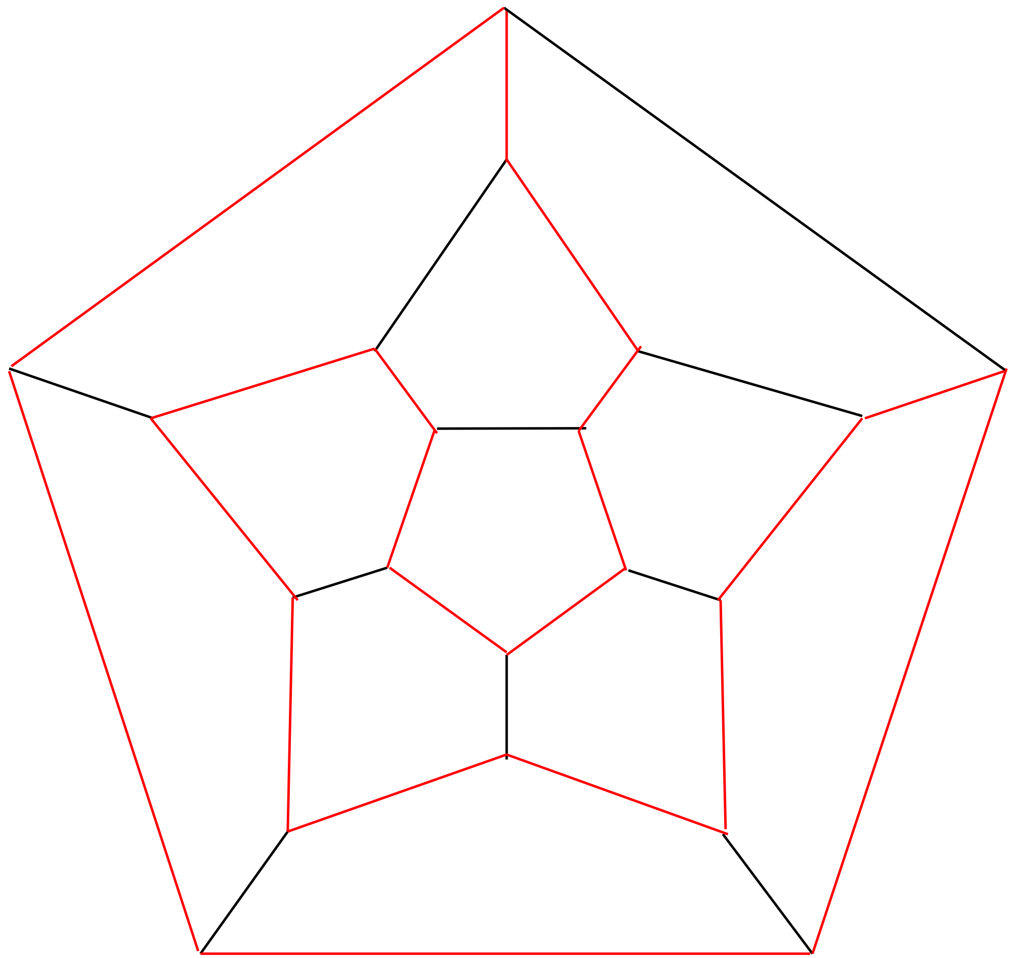
• If a graph has a Hamiltonian cycle, it is called a *Hamiltonian graph*.

• If a graph has a Hamiltonian walk, it is called a *semi-Hamiltonian graph*.

☹There are no known (non-trivial) conditions that would be necessary and sufficient for the existence of a Hamiltoinian cycle or a Hamiltoinian walk.

• In this lecture, only simple graphs are considered.





Theorem (Ore, 1960). Let $G = (V, E)$ be a simple graph, where $|V| = n \geq 3$. If for every two vertices $u, w \in V$ the implication

$$(u, w) \notin E \implies \deg(u) + \deg(w) \geq n$$

holds, then the graph G is Hamiltonian.

Corollary (Dirac, 1952). If $G = (V, E)$ is a simple graph having n vertices and for each $v \in V$ we have $\deg(v) \geq \frac{n}{2}$ then G is a Hamiltonian graph.

Proof of the Corollary. For every two verticesu $u, w \in V$ (whether they are neighbours or not) the inequality $\deg(u) + \deg(w) \geq n$ holds, thus Ore's theorem implies that G is Hamiltonian.

Proof of the theorem. If $n = 3$ then the only graph satisfying the assumption is K_3. It is Hamiltonian.

Let $n \geq 4$. Let the assumption of the theorem hold, but let the conclusion be wrong.

If we add edges to the graph, the assumption will still hold. Add edges to G until we reach the graph G' such that it is not Hamiltonian, but addition of any new vertex would give a Hamiltonian graph.

Let $e = (u, w) \in V \times V$ be an edge not present in G'. The graph $G' \cup \{e\}$ has a Hamiltonian cycle

$$u = v_0 \text{ --- } v_1 \text{ --- } v_2 \text{ --- } \cdots \text{ --- } v_{n-1} = w \overset{e}{\text{ --- }} u \ .$$

Grph G' has a Hamiltonian walk

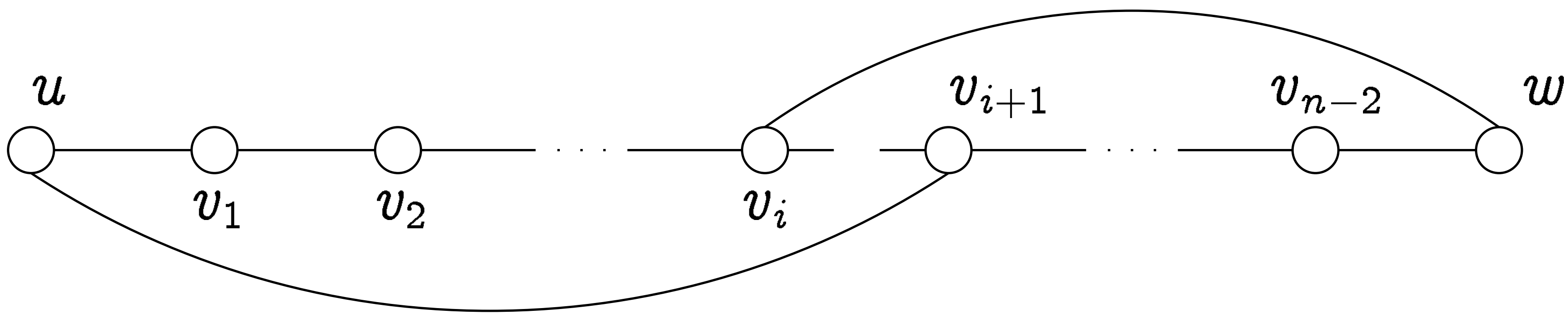
$$P : u = v_0 \text{ --- } v_1 \text{ --- } v_2 \text{ --- } \cdots \text{ --- } v_{n-1} = w \ .$$

This walk has $n - 1$ edges.

Let

- E_u be the set of edges (v_i, v_{i+1}) where $(u, v_{i+1}) \in E$.
- E_w be the set of edges (v_i, v_{i+1}) where $(v_i, w) \in E$.

Using the assumption of the theorem, we get $|E_u| + |E_w| \geq n$. Thus, there is an edge (v_i, v_{i+1}) in the intersection $E_u \cap E_w$. Besides, $i \neq 0$ and $i \neq n-2$, since $(u, w) \notin E$.



We have found a Hamiltonian cycle in G'. $\square$

Theorem (Bondy and Chvátal, 1976). Consider a simple graph $G = (V, E)$ and let $u, v \in V$ be non-neighbouring vertices such that $\deg(u)+\deg(v) \geq |V|$. Then G is Hamiltonian iff $G \cup \{(u, v)\}$ is Hamiltonian.

Proof. The direction "G Hamiltonian $\Rightarrow$ $G \cup \{(u, v)\}$ Hamiltonian" is obvious. Proof of the other direction was given in the proof of Ore's theorem. $\square$

Graph $G = (V, E)$ is called *Ore-closed* if for any two different vertices $u, v \in V$ the implication

$$\deg(u) + \deg(v) \geq |V| \implies (u, v) \in E$$

holds.

Graph $G' = (V, E')$ is called *Ore closure* of graph $G = (V, E)$ and denoted as $\mathcal{O}(G)$ if the following holds:

- G' is Ore-closed;
- $E \subseteq E'$;
- E' is the least possible set with the above properties.

Lemma. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be Ore-closed graphs. Then $G = (V, E_1 \cap E_2)$ is Ore-closed.

Proof. Let $u, v \in V$ and $\deg_G(u) + \deg_G(v) \geq |V|$. Then we have

$$\deg_{G_1}(u) + \deg_{G_1}(v) \geq |V| \text{ and } \deg_{G_2}(u) + \deg_{G_2}(v) \geq |V|,$$

since $\deg_{G_i}(u) \geq \deg_G(u)$ and $\deg_{G_i}(v) \geq \deg_G(v)$.

As G_1 and G_2 are Ore-closed, we get $(u, v) \in E_1$ and $(u, v) \in E_2$, implying $(u, v) \in E_1 \cap E_2$. $\square$

The Lemmma implies that all graphs have Ore closures.

Algorithm (for finding Ore closure). Consider a simple graph $G = (V, E)$.

1. Find $u, v \in V$ such that $\deg(u) + \deg(v) \geq |V|$ and $(u, v) \notin V$. If there are no such vertices, output G and stop.

2. Add the edge (u, v) to E and return to step 1.

Proposition. The result of the algorithm does not depend on the choice of vertices u, v on step 1.

Proof. Assume we can get two different outcomes $G_1 = (V, E \dot\cup E_1)$ and $G_2 = (V, E \dot\cup E_2)$ starting from graph $G = (V, E)$ (so that $E_1 \neq E_2$). W.l.o.g. assume $E_1 \setminus E_2 \neq \emptyset$.

Elements of the set $E_1 \setminus E_2$ are added to the graph G_1 in some order as the algorithm proceeds. Let (u, v) be the first one in this order. Let $E_1' \subseteq E_1$ be thet set of all edges added before the edge (u, v).

We have $E_1' \subseteq E_2$. Thus, in the graph G_2 the condition $\deg(u) + \deg(v) \geq |V|$ holds. A contradiction with the assumption $(u, v) \notin E_2$. $\square$

Theorem. The algorithm finds Ore closure of graph G.

Proof. The proof follows from these four claims:

1. Edge set of the output graph of the algorithm is a superset of the edge set of the input graph.
2. The algorithm is monotone, i.e. if $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, where $E_1 \subseteq E_2$, the algorithm turns them into graphs $G_1' = (V, E_1')$ and $G_2' = (V, E_2')$, where $E_1' \subseteq E_2'$. The proof is similar to the proof of the previous proposition.
3. The output graph of the algorithm is Ore-closed.
4. If the input of the algorith is an Ore-closed graph, the algorithm will output it.

$\square$

Corollary. A graph is Hamiltonian iff its Ore closure is Hmiltonian.

Proof. This is a consequence of the closure finding algorithm and Bondy-Chvátal theorem. □

Corollary. Let $G = (V, E)$ be a simple graph with $|V| = n \geq 3$. If $\mathcal{O}(G) = K_n$ then G is Hamiltonian.

Proof. K_n is Hamiltonian. □

Theorem. Let $G = (V, E)$ be a non-Hamiltonian graph on n vertices. Then there exists $k < \frac{n}{2}$ such that G has k vertices with degree at most k and $n - k$ vertices with degree at most $n - k - 1$.

Proof. Let $\mathcal{O}(G) = (V, E')$. Since $\mathcal{O}(G) \neq K_n$, there exist vertices u and w such that $(u, w) \notin E'$. Take u and w so that the sum $\deg_{E'}(u) + \deg_{E'}(w)$ is maximal.

We have $\deg_{E'}(u) + \deg_{E'}(w) \leq n - 1$, since otherwise $(u, w) \in E'$ (according to the definition of Ore closure). Let

$$U = \{u' \mid u' \neq u, (u, u') \notin E'\}$$
$$W = \{w' \mid w' \neq w, (w, w') \notin E'\} \ .$$

W.l.o.g. assume $\deg_{E'}(u) \leq \deg_{E'}(w)$. Let $k = \deg_{E'}(u)$.

1. $\deg_{E'}(u) + \deg_{E'}(w) \leq n - 1$.
2. $\deg_{E'}(u) + \deg_{E'}(w)$ is the maximal possible.
3. $k = \deg_{E'}(u) \leq \deg_{E'}(w)$.
4. 1. and 3. give $k \leq \frac{n-1}{2} < \frac{n}{2}$.
5. 2. gives $\deg_{E'}(w') \leq \deg_{E'}(u)$ for any $w' \in W$. Besides, $\deg_{E'}(u') \leq \deg_{E'}(w)$ for any $u' \in U$.
6. $|U| = n - 1 - \deg_{E'}(u)$ and $|W| = n - 1 - \deg_{E'}(w)$. This is proven by a simple counting argument.
7. 1. and 6. give $|W| \geq k$.
8. 5. gives $\deg_{E}(w') \leq \deg_{E'}(w') \leq \deg_{E'}(u) = k$ for any $w' \in W$.

We have k vertices with degree $\leq k$.

1. $\deg_{E'}(u) + \deg_{E'}(w) \leq n - 1$.

4. $k \leq \frac{n-1}{2} < \frac{n}{2}$.

5. $\deg_{E'}(u') \leq \deg_{E'}(w)$ for any $u' \in U$.

6. $|U| = n - 1 - \deg_{E'}(u)$.

9. 6. gives $|U| = n - k - 1$. Thus $|U \cup \{u\}| = n - k$.

10. For each $u' \in U$ we get from 5. and 1. that

$$\deg_E(u') \leq \deg_{E'}(u') \leq \deg_{E'}(w) \leq n - 1 - k \ .$$

11. 4. gives $\deg_E(u) \leq \deg_{E'}(u) = k \leq \frac{n-1}{2} \leq n - 1 - k$.

We have $n - k$ vertices with degree $\leq n - k - 1$.

Corollary. Consider a graph $G = (V, E)$ on n vertices such that for each $k < \frac{n}{2}$ the graph has less than k vertices with degree at most k or less than $n-k$ vertices with degree at most $n-k-1$. Then G is Hamiltonian.

Proof. From the previous theorem: $(\mathcal{A} \Rightarrow \mathcal{B}) \Leftrightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$. $\square$

The same claim for degree sequences:

Corollary. Consider a graph $G = (V, E)$ with degree sequence $(a_1, \ldots, a_n)$. If for each $k < \frac{n}{2}$ we have $(a_k \leq k) => (a_{n-k} \geq n-k)$ then G is Hamiltonian.

Call the degree sequence $(a_1, \ldots, a_n)$ *Hamiltonian* if each graph G with degree sequence $(b_1, \ldots, b_n)$ where $b_i \geq a_i$ $(1 \leq i \leq n)$ is Hamiltonian.

Theorem. Degree sequence $(a_1, \ldots, a_n)$ is Hamiltonian iff for each $k < \frac{n}{2}$ we have $(a_k \leq k) \Rightarrow (a_{n-k} \geq n - k)$.

Proof. $\Leftarrow$ is proven in the previous slide

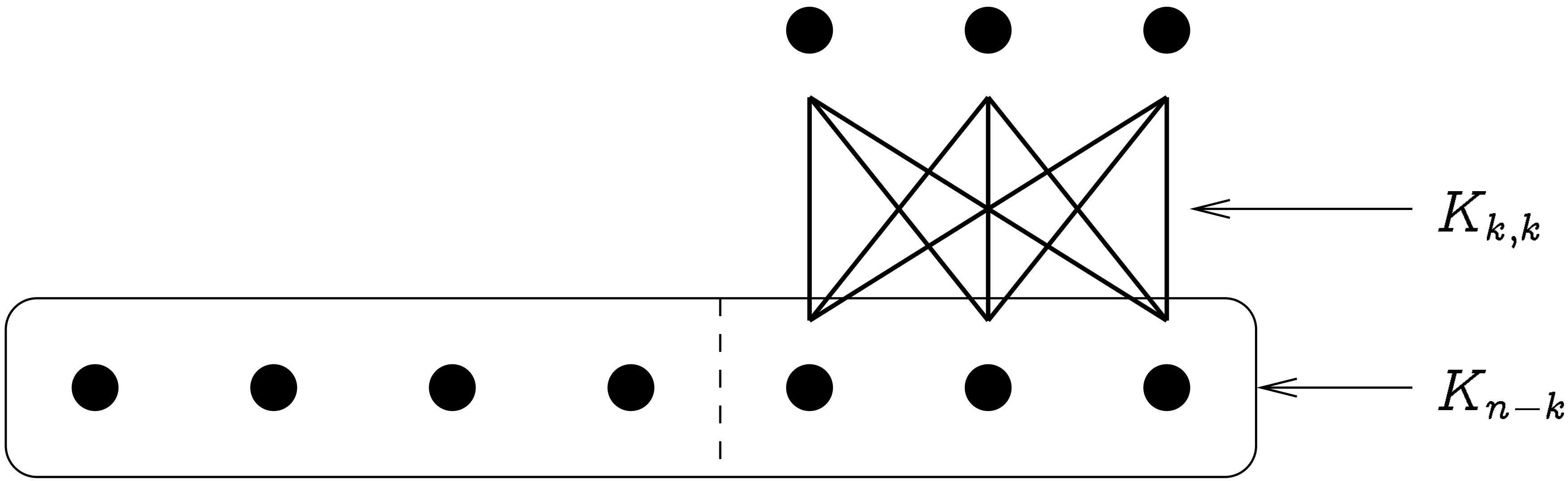
$\Rightarrow$ Assume that $(a_1, \ldots, a_n)$ does not satisfy the required condition. We will construct a graph with degree sequence $\geq (a_1, \ldots, a_n)$ that is not Hamitlonian.

If the condition is not satisfied, we must have a k such that $a_k \leq k$ and $a_{n-k} \leq n - k - 1$.

For a given k the largest such degree sequence is

$$(\underbrace{k,\ldots,k}_{k},\underbrace{n-k-1,\ldots,n-k-1}_{n-2k},\underbrace{n-1,\ldots,n-1}_{k}) \ .$$

A non-Hamiltonian graph with such a degree sequence:



$\square$

Let $G = (V, E)$ be a graph.

- $S \subseteq V$ is an *independent set* if $\forall u, v \in S$, the vertices u, v are not adjacent.
- Define $\alpha(G)$ as the maximum cardinality of an independent set in G.
- Let $k \in \mathbb{N}$. Graph G is *k-connected*, if removal of any $(k-1)$ vertices from G (with incident edges) does not disconnect G.
- Let $k(G)$ be maximum such k, that G is k-connected.

Theorem. If G has at least 3 vertices and $k(G) \geq \alpha(G)$, then G is Hamiltonian.

Theorem. A Graph $G = (V, E)$ is k-connected iff any two vertices can be joined by k vertex-disjoint paths.

We'll give the proof in the 5th lecture.

Theorem. If G has at least 3 vertices and $k(G) \geq \alpha(G)$, then G is Hamiltonian.

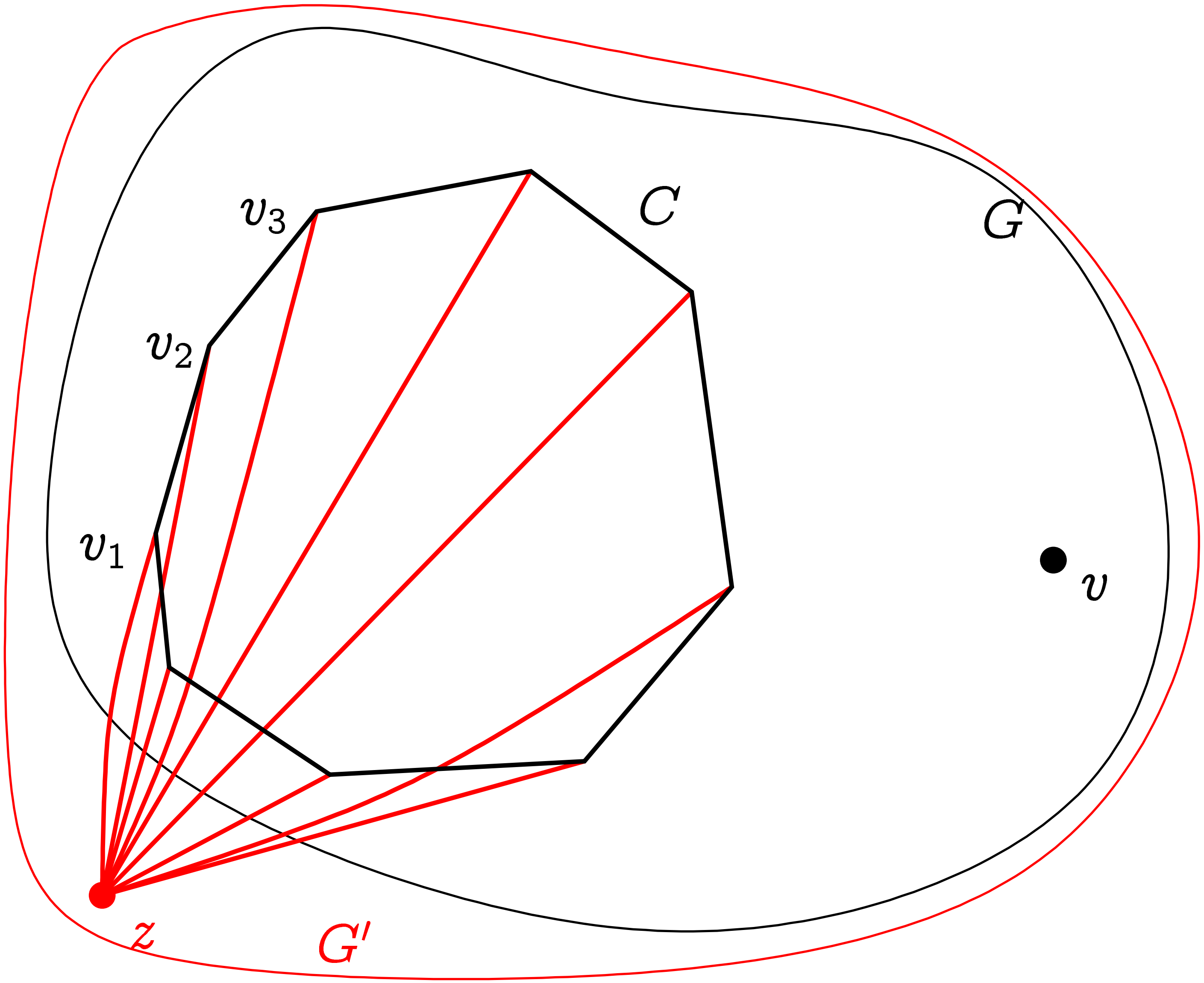
Proof. Let C be a cycle in G of maximum length. Assume the opposite: C is not a Hamiltonian cycle.

Denote: $k = k(G)$, $n = |V(C)|$, $\{v_1, \ldots, v_n\} = V(C)$, numbered along the cycle.

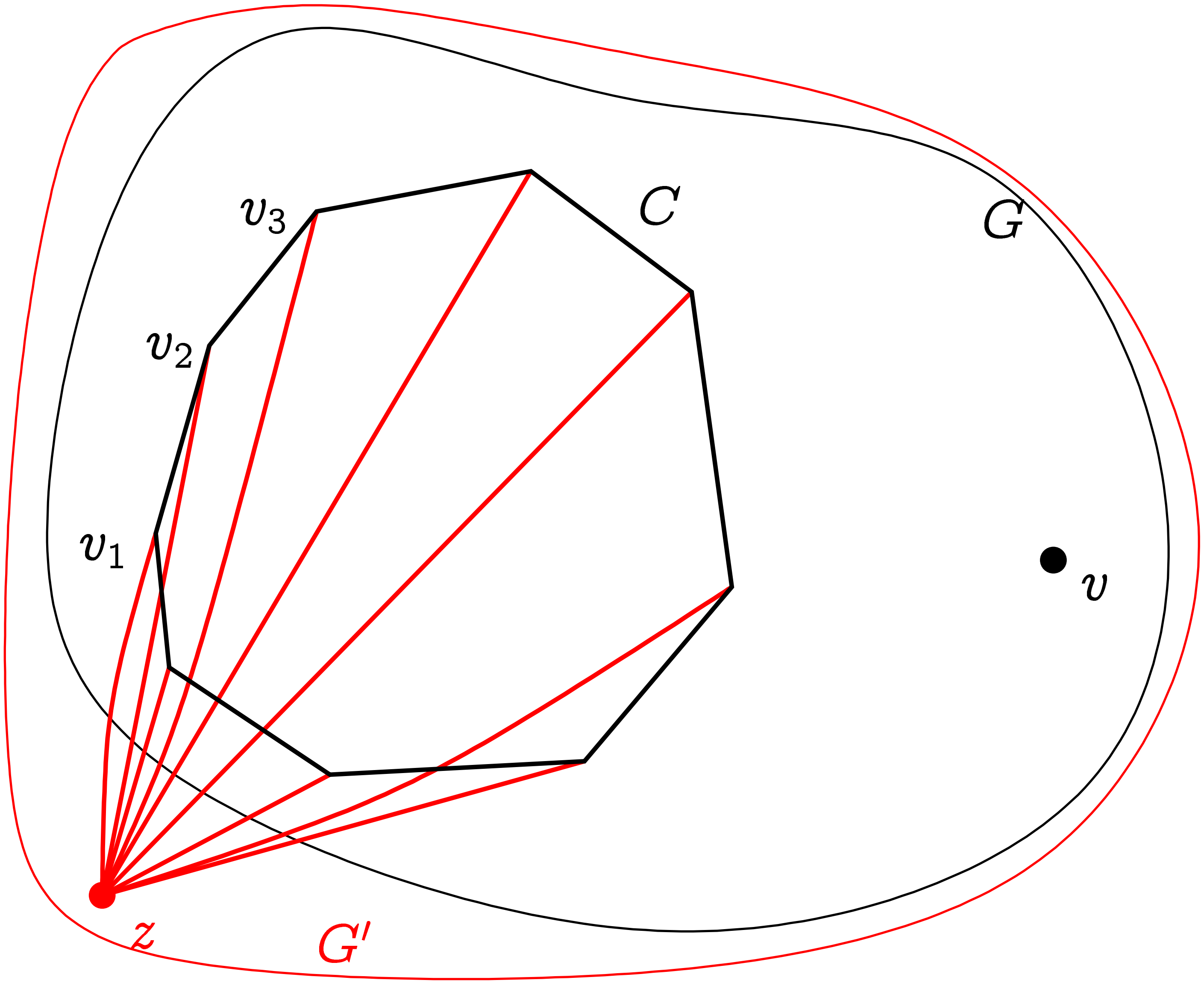
$n \geq k + 1$. (Why?)

Let $v \in V(G) \backslash V(C)$.

Let z be a new vertex, adjacent to all vertices of C. Let the resulting graph be G'.



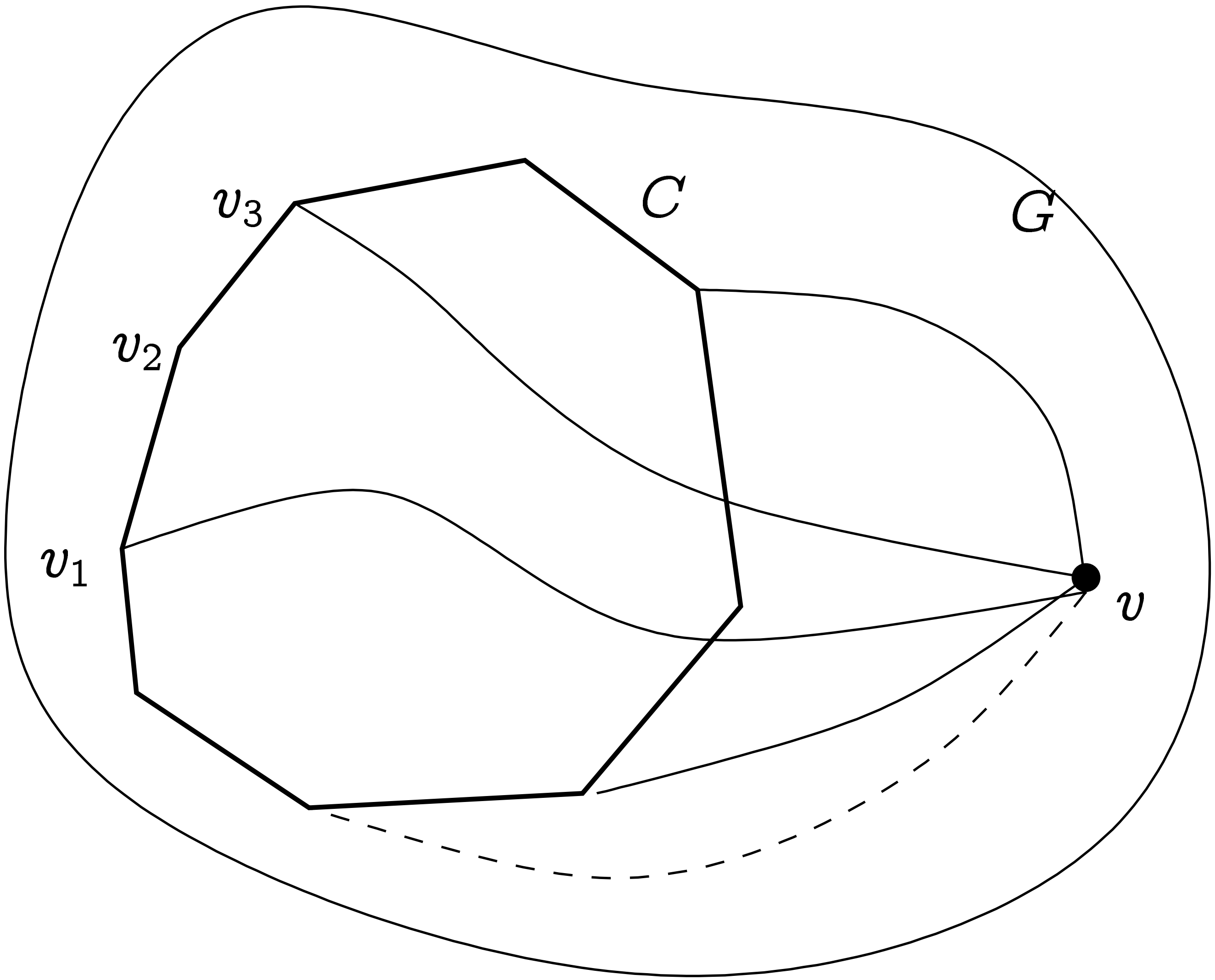
G' is still k-connected. (Why?)

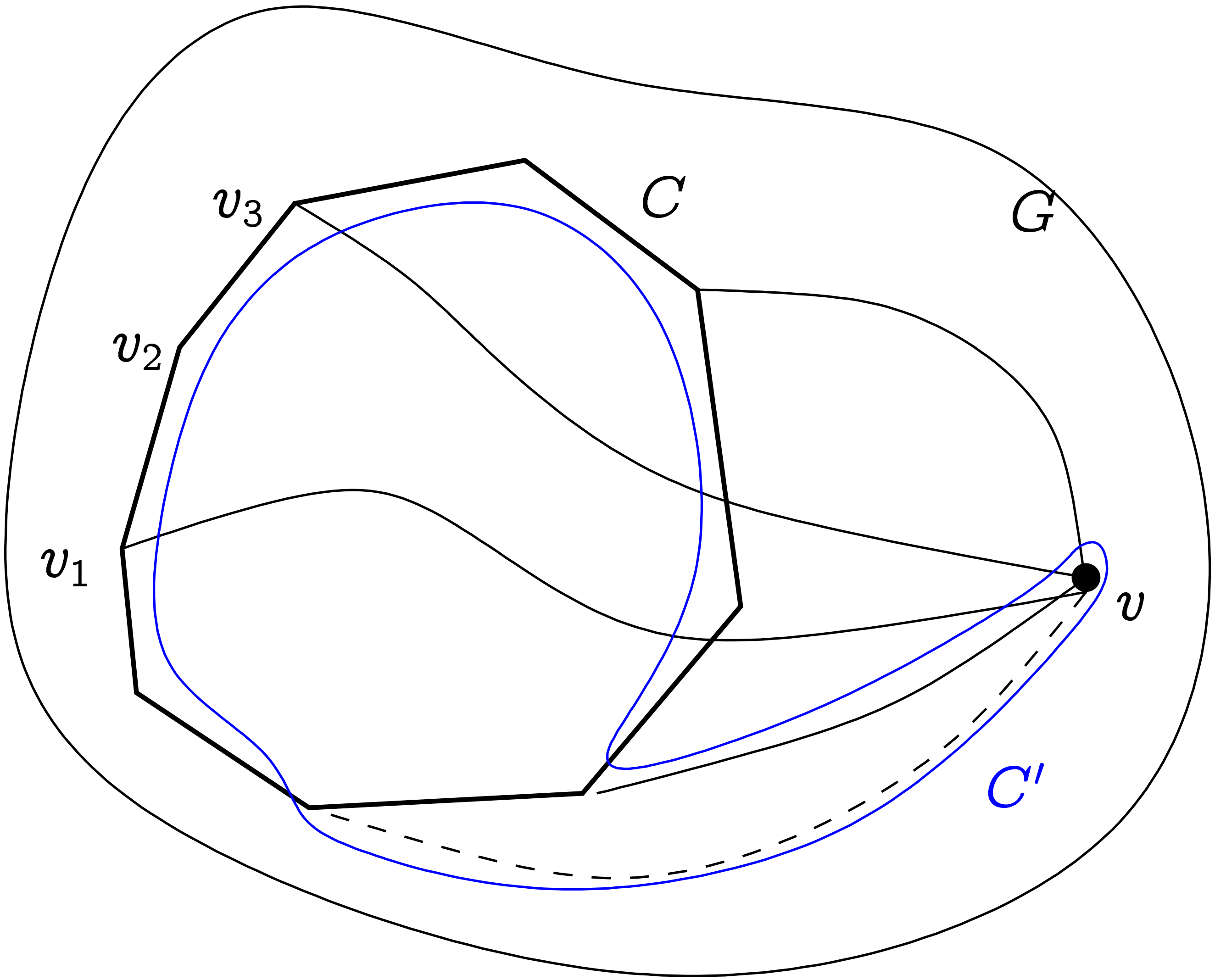


There are k vertex-disjoint paths from v to vertices $\{v_1, \ldots, v_n\}$.

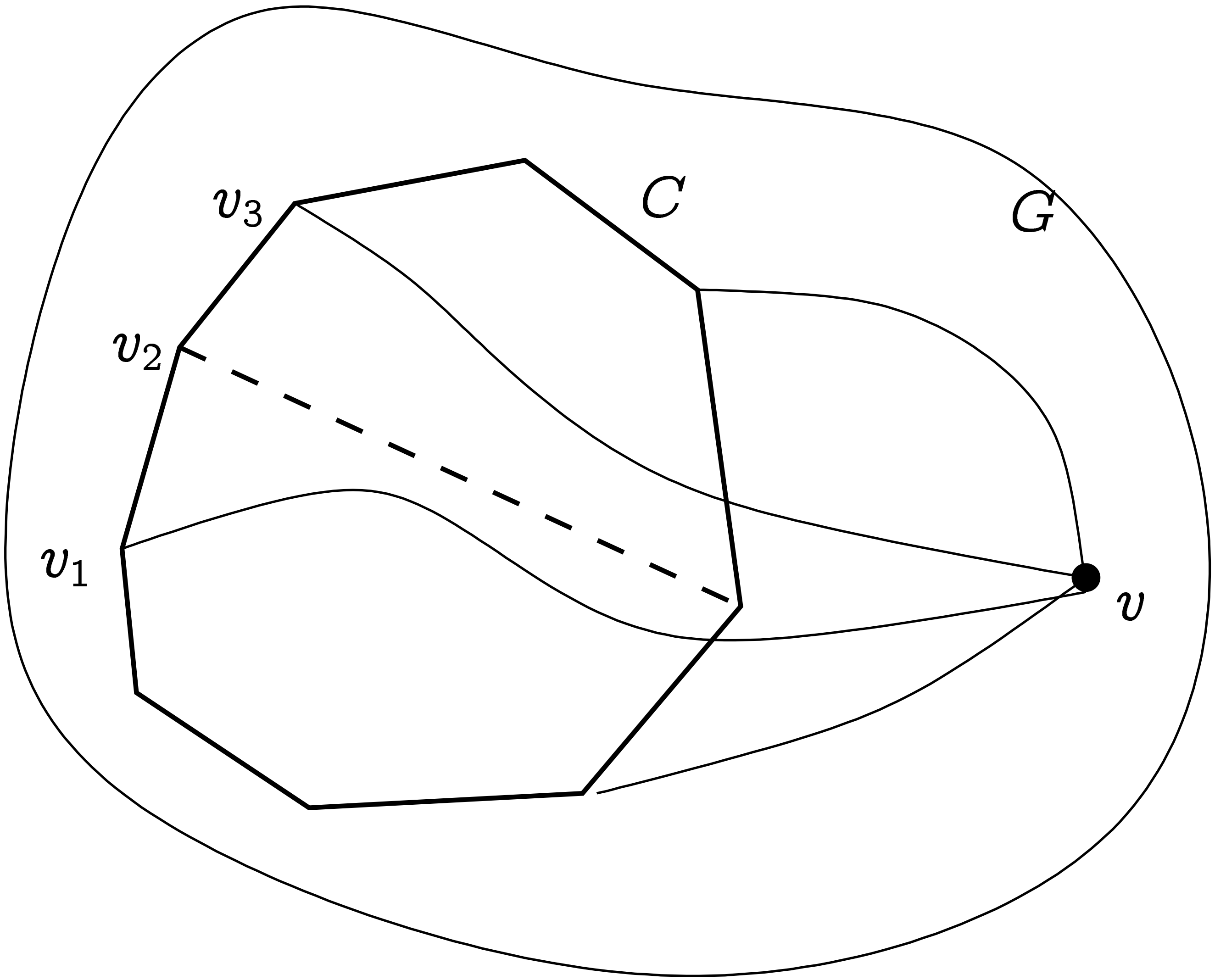
If there is a path from v to v_i, then there is no path from v to v_{i+1}.

(indices *modulo* n)

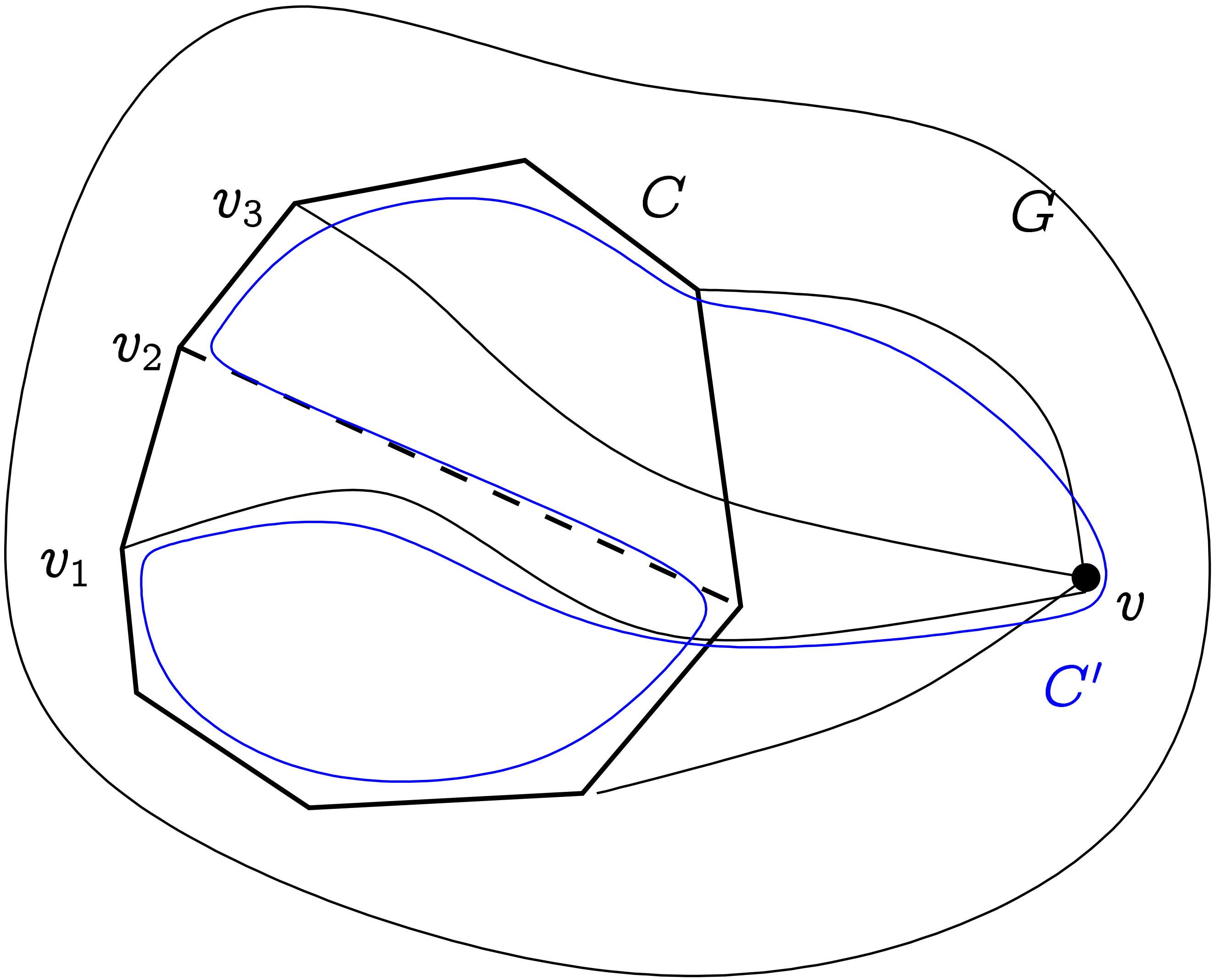

v_3
C
G
v_2
v_1
v


v_3
C
G
v_2
v_1
v
C'

If there is a path from v to v_i, and from v to v_j, then there is no edge between v_{i+1} and v_{j+1}.



v_3
C
G
v_2
v_1
v



v_3
C
G
v_2
v_1
v
C'

Let the paths be from v to $v_{i_1}, \ldots, v_{i_k}$.

Consider the set $S = \{v, v_{i_1+1}, \ldots, v_{i_k+1}\}$.

S is an independent set and

$|S| = k + 1 > k = k(G) \geq \alpha(G) \geq |S|$.

Contradiction. $\square$