MTAT.07.004 — Complexity Theory September 22th, 2011

Lecture 4

Lecturer: Peeter Laud *Scribe(s): Ilya Kuzovkin*

Introduction

During this lecture we looked at the proof of the halting theorem, which shows what is halting problem about. After that we prove two theorems (for deterministic and non-deterministic cases) which say that for every two problems f and g there is problem p, which takes more time than f, but less than g. Proved existence of a problem which is not NP-complete (in case if $P \neq NP$). Talked about NP-intermediate problems and looked into some philosophical notes about P complexity.

The Halting Problem

Theorem 1 *If we define language*

$$HALT = \{\langle \alpha, x \rangle | M_\alpha \text{ stops on input } x\}$$

then this language is not accepted by any Turing Machine.

Proof Assume there is TM M_{HALT} which accepts language HALT. Let there be another Turing Machine M' which takes x as input and invokes $M_{\text{HALT}}(\langle x, x \rangle)$ (Run machine encoded by x with input x).

We define M' behaviour as follows: if M_{HALT} accepts, then M' will work indefinitely, if M_{HALT} rejects then M' will return 1.

Now let us say β is encoding of machine M' and we run $M'(\beta)$. We have two cases:

$$M'(\beta) \text{ will stop} \Leftrightarrow M_{\text{HALT}}(\langle \beta, \beta \rangle) \text{ rejects} \Leftrightarrow$$

$$\Leftrightarrow M_\beta(\beta) \text{ will not stop} \equiv M'(\beta) \text{ will not stop}$$

Contradiction!

And second case

$$M'(\beta) \text{ will not stop} \Leftrightarrow M_{\text{HALT}}(\langle \beta, \beta \rangle) \text{ accepts} \Leftrightarrow$$

$$\Leftrightarrow M_\beta(\beta) \text{ will stop} \equiv M'(\beta) \text{ will stop}$$

Contradiction!

■

Deterministic Time Hierarchy Theorem

Theorem 2 *Let f and g be two time-constructible functions, such that $f(n) > n$ and $\lambda n.f(n) \log f(n) \in o(g)$. Then DTIME(f) $\subsetneq$ DTIME(g)*

In other words there always is a function p which takes more time than f, but less than g.

Proof We introduce new function h such that it is more complex that f, but less complex than g:

- $h \in \omega(f)$
- $h(n) \log h(n) \in O(g)$

We define language D

$$D = \{\alpha \in \{0,1\}^* \mid \text{ accepts } \alpha \text{ in } \leq h(|\alpha|) \text{ steps}\}^c$$

Note that ^{c} here means *complementary* – D consist of all such α which do not satisfy the condition in the brackets.

We pick languge $L \in \text{DTIME}(f)$, and let machine M accept that language in time $c \cdot f$
We pick α such than $M_\alpha = M$ and $\frac{h(|\alpha|)}{f(|\alpha|)} > c$.

$$\text{If } \alpha \in L \Rightarrow$$
$$\Rightarrow M_\alpha(\alpha) \text{ accepts with } \leq c \cdot f(|\alpha|) \text{ steps } \Rightarrow$$
$$\Rightarrow M_\alpha(\alpha) \text{ accepts with } \leq h(|\alpha|) \text{ steps } \Rightarrow$$
$$\Rightarrow \text{ from the definition of D we can see that } \alpha \notin D \Rightarrow$$
$$L \neq D$$

At the same time $D \in DTIME(g)$ because, as if follows from the definition of D, we can accept or reject α in time $\leq DTIME(h) \leq DTIME(g)$ ■

Non-deterministic Time Hierarchy Theorem

Theorem 3 *Let f and g be two time-constructible functions, such that $f(n) > n$ and $\lambda n.f(n+1) \in o(g)$. Then NTIME(f) $\subsetneq$ NTIME(g)*

This theorem state the same fact as previous one, but for non-deterministic time. We cannot construct the proof in the same way as the previous one because there we had, that then machine rejects in accept on the complementary set of α. When dealing with non-deterministic machines we cannot say that – if one branch reject, the α can be still accepted in some other branch.

Proof We introduce two new functions h and h' so that $f(n+1) \in o(h')$, $h' \in o(h)$, $h \in o(g)$, or we can say that in terms of time complexity $f \leq h' \leq h \leq g$.
Also we define a new function φ as follows:
$\varphi(1) = 2$
$\varphi(i+1) = 2^{h(\varphi(i))}$
Let $\tilde{\varphi}(n) = max\{i \mid \varphi(i) \leq n\}$ (provide the argument i, with which φ value is closest to n from the left).
Function φ is used to split the set of natural numbers (or: lengths of bit strings) into sets, such that each next set exponentially larger then previous.
Next we define language D:

$$D = \left\{ 1^n \;\middle|\; \begin{array}{l} i := \tilde{\varphi}(n) - 1 \\ n \neq \varphi(i+1) \text{ and } M_i \text{ accepts } 1^{n+1} \text{ in time } h'(n) \\ OR \\ n = \varphi(i+1) \text{ and } M_i \text{ rejects } 1^{\varphi(i)+1} \text{ in time } g(\varphi(i)+1) \end{array} \right\}$$

This can be represented with following figure:

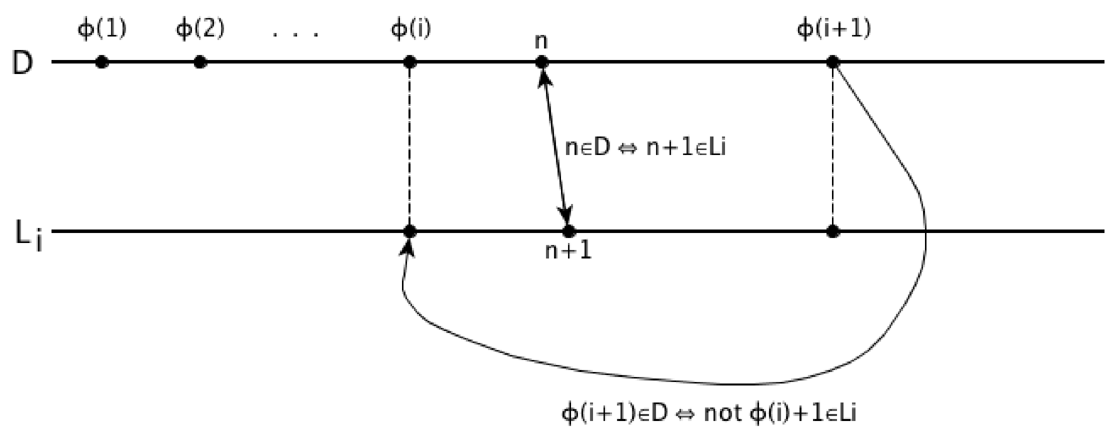


Figure 1: Representation of definition of D

We have machine M_i which accepts language L_i in h' steps.
The rule is that element $n \in D$ iff $n+1 \in L_i$, where i is argument of φ.
And one additional rule says that $\varphi(i+1) \in D$ iff $\varphi(i)+1 \notin L_i$

We want to show that $D \notin \text{NTIME}(f)$ and $D \in \text{NTIME}(g)$

D $\in$ NTIME(g)

First we compute $\varphi(1), \varphi(2), ...$ until n in order to find $\tilde{\varphi}(n)$. We can do it in such naive way since it will take only logarithmic time.
If $n \neq \varphi(i+1)$ when just simulate M_i and see if $n+1 \in L_i$. This is doable in h' time.

The situation is more complicated when $n = \varphi(i+1)$. In this case we have to search through all possible computational paths of M_i to make sure it rejects on every path (only then we can accept n to D). Here comes in play that every $\varphi(i+1)$ is exponentially larger than $\varphi(i)$. This gives us time to compute all $O(2^{g(\varphi(i)+1)})$ paths of M_i. Since both cases are doable in time less than g (by definition of D) we can say that $D \in \mathrm{NTIME}(g)$.

D $\notin$ NTIME(f)

Let language $L \in \mathrm{NTIME}(f)$. L is accepted by machine M_i. Now we assume that $L = D$.

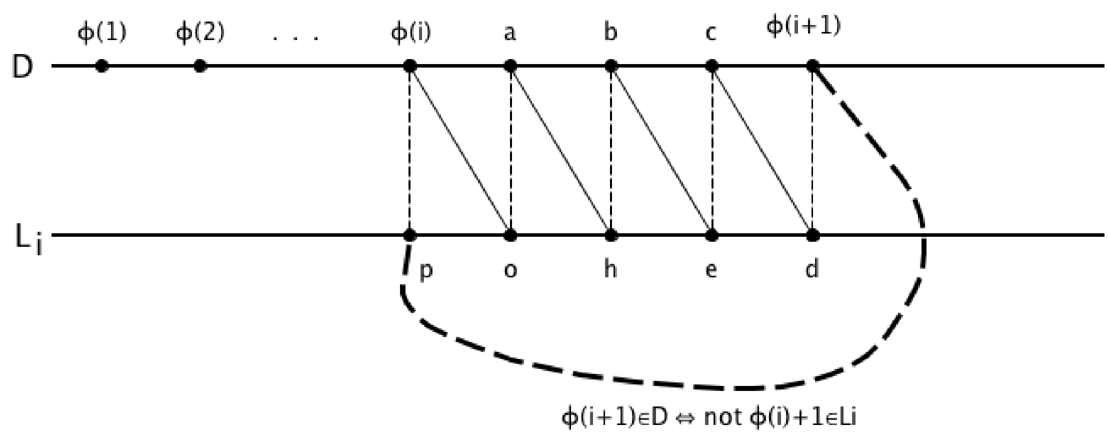


Figure 2: Illustration of contradiction

Now, when we assumed that $L = D$ we have three rules:

1. $n \in D$ iff $n+1 \in L_i$ (by definition) – solid line on the Figure 2
2. $\varphi(i+1) \in D$ iff $\varphi(i)+1 \notin L_i$ (by definition) – bold dashed line on the Figure 2
3. $n \in D$ iff $n+1 \in L_i$ (new rule comes from $L = D$) – dashed line on the Figure 2

Now if we apply these rules one by one we see:

If $p \in L_i \Rightarrow^{\text{(rule 3)}} \varphi(i) \in D \Rightarrow^{\text{(rule 1)}} o \in L_i \Rightarrow^{\text{(rule 3)}} a \in D \Rightarrow^{\text{(rule 1)}} \ldots \Rightarrow^{\text{(rule 3)}} \varphi(i+1) \in D \Rightarrow^{\text{(rule 2)}} p \notin L_i \Rightarrow$ Contradiction!

In the same way the second case

If $p \notin L_i \Rightarrow^{\text{(rule 3)}} \varphi(i) \notin D \Rightarrow^{\text{(rule 1)}} o \notin L_i \Rightarrow^{\text{(rule 3)}} a \notin D \Rightarrow^{\text{(rule 1)}} \ldots \Rightarrow^{\text{(rule 3)}} \varphi(i+1) \notin D \Rightarrow^{\text{(rule 2)}} p \in L_i \Rightarrow$ Contradiction!

We have showed $D \notin \mathrm{NTIME}(f)$.
Now we know that $D \in \mathrm{NTIME}(g)$ and $D \notin \mathrm{NTIME}(f)$, which means $\mathrm{NTIME}(f) \subsetneq \mathrm{NTIME}(g)$. ■

Existence of not NP-complete problems

Theorem (Ladner) 4 *If $P \neq NP$ then there exists a language $A \in NP \backslash P$ that is not NP-complete.*

Proof We will prove by constructing such language A. First we define a few things.

- $M_1, M_2, ...$ will be polynomial-time DTM such that language L_i is accepted by machine M_i
- $f_1, f_2, ...$ will be polynomial-time computable functions such that M_i computes f_i in time $O(n^i)$

We define A as

$$A = \{x \in \{0,1\}^* \mid x \in \text{ SAT and } g(|x|) \text{ is even}\}$$

The function g is defined below.
Function $g(n)$ is defined as follows:
$g(0) = 2$
$g(1) = 2$
For $n \geq 2$ we do following recursive iterations:

1. take u to be largest such that $g(u)$ was computed
2. $k = g(u)$
3. $i = \lfloor \frac{k}{2} \rfloor$
4. for $j \in \{0, 1, 2, ...\}$
   (a) If k is even check $B_j \in L_i$ **XOR** $B_j \in A$
   (b) If k is odd check $B_j \in SAT$ **XOR** $f_j(B_j) \in A$
   (c) Now look at **XOR** result – if it is $true$ return $k+1$, if $false$ return k

Note: since function itself does not have any boundaries and will work indefinitely, we will use a counter in the parallel process, which will stop the execution of the computation after n time units (for example seconds) have passed.

We will go through three claims which together show that A is in NP, but is not in P and is not in NP-complete.

A $\in$ NP

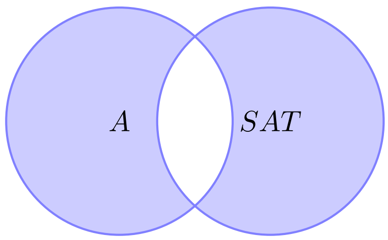
To accept or reject we have to check if x is in SAT. This is NP problem. We also have to compute $g(|x|)$, which can be done in time $O(|x|)$. so A belongs to NP by definition.

A $\notin$ P

Assume A $\in$ P. In that case there is language $L_i = A$. Let us consider smallest such i.
In that case $g(n)$ will never be more that $2i$.

If during the iterations $g(n)$ will be most of the time $2i$ it will mean that **XOR** after some point will become always $false$ $\Rightarrow$ it is stuck in the "k is even" branch (step 4.a) $\Rightarrow$ the result in "k is odd" will give us only finite set for SAT:

$A \ \Delta \ SAT$



And because A $\in$ P and SAT differs from A only for a finite number of bit-strings x, we can compute SAT in polynomial time $\Rightarrow$ Contradiction!

On the other hand if $g(n)$ has not reached $2i$ and most of the time equals $2i+1$ it will mean that iterations are stuck in the "k is odd" branch (step 4.b) $\Rightarrow$ **XOR** gives $false$ $\Rightarrow$ There is $B \in SAT$ and $f(B) \in A \Rightarrow$ SAT $\leq_m^p$ A $\Rightarrow$ since A $\in$ P then due to reducibility SAT is also in P $\Rightarrow$ Contradiction!

SAT is not reducible to A

If we show that SAT is not reducible to A we will show that A is not in NP-complete.
Assume SAT is reducible to A. Then there should be such f_i that $f_i(SAT) = A$.
We in the same way as in the previous section: $g(n)$ never grows past $2i+1$.

If $g(n) = 2i+1$ most of the time, then it is stuck in the branch 4.b and A will be finite $\Rightarrow$ A $\in$ P $\Rightarrow$ SAT $\in$ P $\Rightarrow$ Contradiction!

And another case if $g(n) = 2i$ most of the time $\Rightarrow$ stuck in 4.a branch $\Rightarrow$ there is $L_i = A$ $\Rightarrow$ since $L_i \in$ P then also A $\in$ P $\Rightarrow$ SAT $\in$ P $\Rightarrow$ Contradiction!

$\blacksquare$

NP-intermediate problems

Problems which are in NP, but are not in P or NP-complete are called NP-intermediate. There is no proof of existence of such (otherwise it would mean that P $\neq$ NP), but there are several problems which are considered to be a good candidates to be NP-intermediate:

- Finding whether two graphs are isomorphic
- Integer factorisation
- Discrete logarithm problem

Note about polynomial complexity

In practice, we are interested in the complexity class P because our experience shows that if we find a solution to some practically significant problem in time $p(n)$, where p is a polynomial and n is the size of the problem instance, then we eventually also find a solution that works in time $q(n)$, where q is a polynomial with a small degree. Such solution is practical and so we think of the complexity class P as the class of "problems solvable in practice".

In theory, it is not the case that for any polynomial-time solution we'll find an equivalent one that only has a small degree. As a simple application of the deterministic time hierarchy problem, $\mathsf{DTIME}(\lambda n.n^{99}) \subsetneq \mathsf{DTIME}(\lambda n.n^{100})$. Hence there exists a problem that is solvable in time $O(n^{100})$ (completely infeasible in practice), but is not solvable in time $O(n^{99})$. We can only assume that these problems do not have practical significance.