Universal Composability
alias
Reactive Simulatability

On security definitions

- Real vs. ideal functionality...
- The ideal functionality for computing the function f with n inputs and outputs:
  - Parties $P_1, \ldots, P_n$ hand their inputs $x_1, \ldots, x_n$ over to the functionality.
  - The ideal functionality computes $(y_1, \ldots, y_n) = f(x_1, \ldots, x_n)$.
    - ...tossing coins if f is randomized.
  - The ideal functionality sends y_i to P_i.

Ideal functionality $MPC_n^{\textbf{Ideal}}$

- Has n input ports and n output ports.
- Initial state: $x_1, \ldots, x_n$ are undefined.
- On input (input, v) from port $in_i?$:
  - If x_i is defined, then do nothing.
  - If x_i is not defined, then set $x_i := v$.
- If $x_1, \ldots, x_n$ are all defined then compute $(y_1, \ldots, y_n)$.
- For all i, write y_i to port $out_i!$.

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- For all i, write y_i to port $out_i!$.

How do we run it (connections, scheduling)? What it means for a party to be corrupted?

Real functionality $MPC_n^{\textbf{Real}}$

- Conceptually made up of n identical machines P_i.
  - Has ports $in_i?$, $out_i!$, network ports...
- Initialization: P_i learns his name i.
- On input (input, v) from port $in_i?$ put $x_i := v$ and start executing the MPC protocol...
- If the protocol has finished execution then write y_i to $out_i!$.

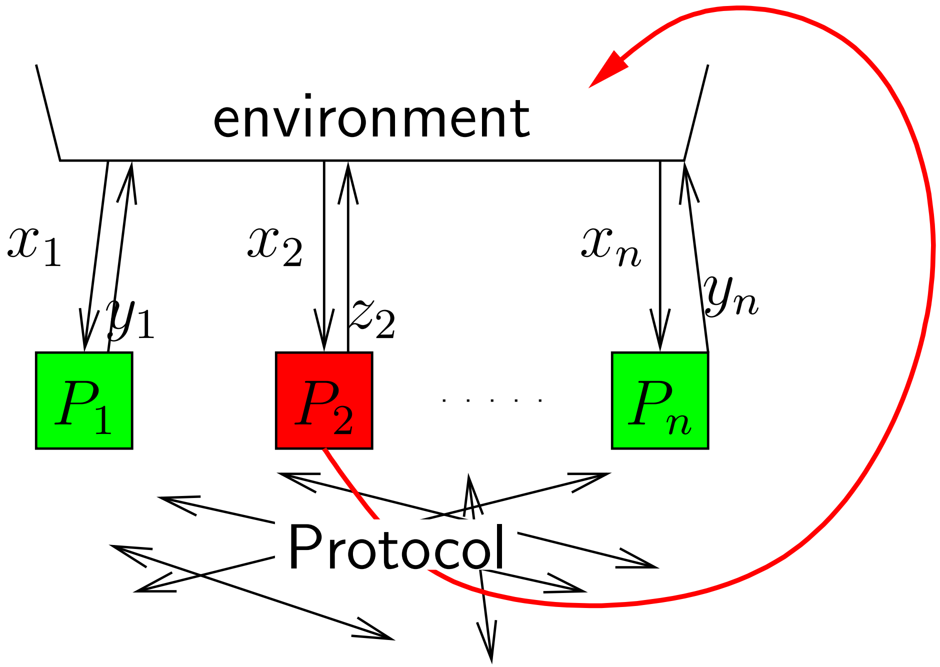
Real functionality $MPC_n^{\textsf{Real}}$

- Conceptually made up of n identical machines P_i.
  - Has ports $in_i?$, $out_i!$, network ports...
- Initialization: P_i learns his name i.
- On input (input, v) from port $in_i?$ put $x_i := v$ and start executing the MPC protocol...
- If the protocol has finished execution then write y_i to $out_i!$.

- Cannot speak about the indistinguishability of $MPC^{\textsf{Ideal}}$ and $MPC^{\textsf{Real}}$ because the set of ports is different.
  - We have to simulate something...

Reactive functionalities

- MPC^{Ideal} worked like this:
  - Get the inputs
  - Give the outputs
- MPC^{Ideal} is non-reactive.
- A reactive functionality gets some inputs, produces some outputs, gets some more inputs, produces some more outputs, etc.
  - Example: secure channel from A to B.
- Further inputs may depend on the previous outputs.
  - Or on the messages sent during the processing of previous inputs.


environment
x_1
y_1
x_2
z_2
x_n
y_n
P_1
P_2
P_n
. . . . . .
Protocol

Composability

- We want the real functionality to be usable instead of the ideal one.
- In all possible contexts!
- $\mathcal{F}^{\mathsf{real}}$ may make no behaviour possible that is impossible with $\mathcal{F}^{\mathsf{ideal}}$.
- Recall the last two exercises of the last test in Cryptology I. . .

Probabilistic I/O automata

A PIOA M has

- The set of possible states Q^M;
- The initial state $q_0^M \in Q^M$ and final states $Q_F^M \subseteq Q^M$;
- The sets of ports:
  - input ports $\mathbf{IPorts}^M$,
  - output ports $\mathbf{OPorts}^M$,
  - clocking ports $\mathbf{CPorts}^M$;
- A probabilistic transition function δ^M:
  - domain: $Q^M \times \mathbf{IPorts}^M \times \{0,1\}^*$;
  - range: $Q^M \times (\mathbf{OPorts}^M \to (\{0,1\}^*)^*) \times (\mathbf{CPorts}^M \cup \{\perp\})$

  ... in our examples implemented by a PPT algorithm.

  - Q^M, Q_F^M and q_0^M may (uniformly) depend on the security parameter.

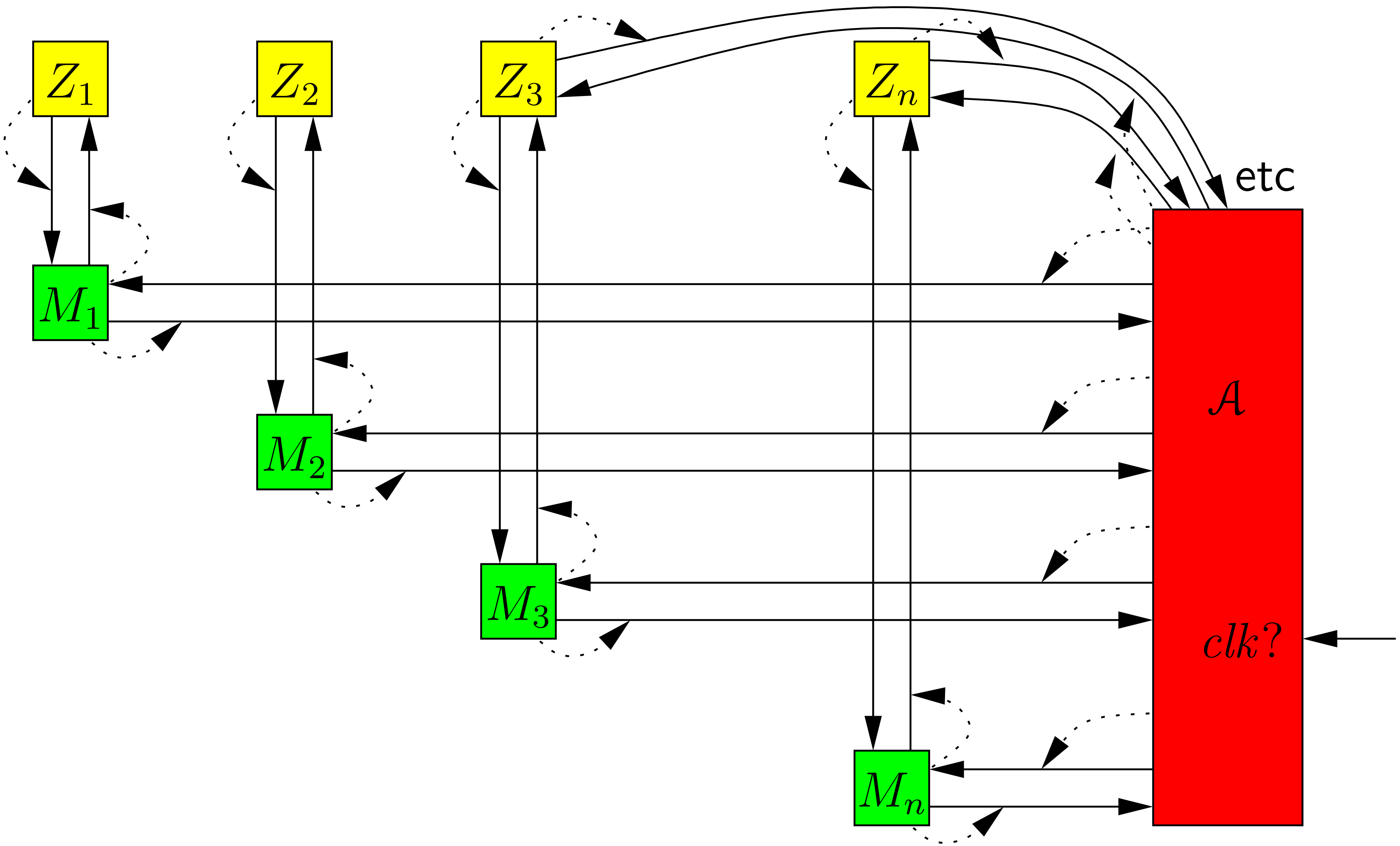
A transition of a PIOA

- The type of δ^M tells us some things about an execution step of a PIOA:
  - Input: one message from one of the input ports.
  - Output: a list of messages for each of the output ports.
  - Also output: a choice of zero or one clocking ports.
- The internal state may change, too.

Channels and collections

- A set $\mathbf{Chans}$ of channel names is given.
- There is a distinguished $clk \in \mathbf{Chans}$, representing global clock.
- For a channel c, its input, output and clocking ports are $c?$, $c!$ and $c^{\triangleleft}!$.
- A closed collection C is a set of PIOAs, such that
  - no port is repeated;
  - For each $c \in \mathbf{Chans} \backslash \{clk\}$ occurring in C: the ports $c?$, $c!$ and $c^{\triangleleft}!$ are all present.
  - $clk?$ is present. $clk!$ and $clk^{\triangleleft}!$ are not present.
- A collection C is a set of PIOAs that can be extended to a closed collection.
  - Let freeports(C) be the set of ports that the machines in C' certainly must have for $C \cup C'$ to be a closed collection.

Example closed collection



Internal state of a closed collection

The state of a closed collection C consists of

- the states of all PIOA-s in C;
  - Initially q_0^M for all $M \in C$.
- the message queues of all channels c in C;
  - I.e. sequences of (still undelivered) messages.
  - Initially the empty queues for all $c \in C$.
- the currently running PIOA M, its input message v and channel c.
  - Initially X, ε and clk, where X is the machine with the port $clk?$.

Execution step of a closed collection

- Invoke the transition function of M with message v on input port $c?$.
  - Update the internal state of M.
  - If $(v_1, \ldots, v_k)$ was written to port $c'!$ then append $v_1, \ldots, v_k$ to the end of the message queue of c'.
- If M is X and it reached the final state then stop the execution.
- Otherwise, if M picked a clock port $c'^{\triangleleft}!$ and the queue of c' is not empty, then define the new (M, v, c):
  - c is c';
  - v is the first message in the queue of c', which is removed from the queue;
  - M is the machine with the port $c'?$.
- Otherwise set $(M, v, c) := (X, \varepsilon, clk)$.

Trace of the execution

Each execution step adds a tuple consisting of

- the machine that made the step;
- the incoming message and the channel;
- the random coins that were generated and the new state and messages that were produced.

to the end of the trace so far.

The semantics of a closed collection is a probability distribution over traces (for a given security parameter).

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The semantics of a closed collection is a probability distribution over traces (for a given security parameter).

Given trace tr and a set of machines $\mathcal{M}$, the restriction of the trace $tr|_{\mathcal{M}}$ consists of only those tuples where the machine belongs to $\mathcal{M}$.

Combining PIOAs

The combination of PIOAs $M_1, \ldots, M_k$ is a PIOA M with

- the state space $Q^M = Q^{M_1} \times \cdots \times Q^{M_k}$;
- initial state $q_0^M = (q_0^{M_1}, \ldots, q^{M_k})$;
- final states $Q_F^M = \bigcup_i Q^{M_1} \times \cdots \times Q^{M_{i-1}} \times Q_F^{M_i} \times Q^{M_{i+1}} \times \cdots \times Q^{M_k}$;
- ports $\mathbf{XPorts}^M = \bigcup_i \mathbf{XPorts}^{M_i}$ with $\mathbf{X} \in \{\mathbf{I}, \mathbf{O}, \mathbf{C}\}$;
- Transition function δ^M, where $\delta^M((q_1, \ldots, q_k), c?, v)$ is evaluated by
  - Let i be such that $c? \in \mathbf{IPorts}^{M_i}$.
  - Evaluate $(q_i', f_i, p) \leftarrow \delta^{M_i}(q_i, c?, v)$.
  - Output $((q_1, \ldots, q_{i-1}, q_i', q_{i+1}, \ldots, q_k), f, p)$, where

$$f(c'!) = \begin{cases} f'(c'!), & \text{if } c'! \in \mathbf{OPorts}^{M_i} \\ \varepsilon, & \text{otherwise.} \end{cases}$$

Exercise. How does the semantics of a closed collection change if we replace certain machines in this collection with their combination?

Security-oriented structures

- A structure consists of
  - a collection C;
  - a set of ports $\mathsf{S} \subseteq \text{freeports}(C)$.
    - C offers the intended service on S.
    - The ports $\text{freeports}(C) \setminus \mathsf{S}$ are for the adversary.
- A system is a set of structures.
- A configuration consists of a structure (C, S) and two PIOA-s H and A, such that
  - H has no ports in $\text{freeports}(C) \setminus \mathsf{S}$,
  - $C \cup \{H, A\}$ is a closed collection.
- Let $\mathbf{Confs}(C, \mathsf{S})$ be the set of pairs (H, A), such that (C, S, H, A) is a configuration.

Exercise. What parts of (C, S) determine $\mathbf{Confs}(C, \mathsf{S})$?

Reactive simulatability

- Let (C_1, S) and (C_0, S) be two structures.
- (C_1, S) is at least as secure as (C_0, S) if
  - for all H,
  - for all A, such that $(H, A) \in \mathbf{Confs}(C_1, \mathsf{S})$
  - exists S, such that $(H, S) \in \mathbf{Confs}(C_0, \mathsf{S})$

  such that $[\![C_1 \cup \{H, A\}]\!]|_H \approx [\![C_0 \cup \{H, S\}]\!]|_H$.
- We also say that (C_0, S) simulates (C_1, S).
- The simulatability is universal if the order of quantifiers is $\forall A \exists S \forall H$.
- The simulatability is black-box if
  - there exists a PIOA Sim, such that
  - for all $(H, A) \in \mathbf{Confs}(C_1, \mathsf{S})$ holds

$$(H, A \| Sim) \in \mathbf{Confs}(C_0, \mathsf{S}) \text{ and } [\![C_1 \cup \{H, A\}]\!]|_H \approx [\![C_0 \cup \{H, A, Sim\}]\!]|_H.$$

Exercise. Show that universal and black-box simulatability are equivalent (if the port names do not collide).

Simulatability for systems

- A system Sys_1 is at least as secure as a system Sys_0 if for all structures $(C_1, \mathsf{S}) \in Sys_1$ there exists a structure $(C_0, \mathsf{S}) \in Sys_0$, such that (C_1, S) is at least as secure as (C_0, S).

What is simulatable by what?

- Input $n \in \mathbb{Z}$ from the user, output n to the adversary.
- Input n from the user, output n^2 to the adversary.
- Input n from the user, output $|n|$ to the adversary.
- "Unique identifiers" (randomly generated) from a single party.
- "Unique identifiers" (randomly generated) from k parties.

Example: secure channels for n parties

- Ideal PIOA $\mathcal{I}$ has ports $in_i?$ and $out_i!$ for communicating with the i-th party.
- Input (j, M) on $in_i?$ causes (i, M) to be written to $out_j!$.
- Should model API calls, hence it also has the ports $out_i{}^{\triangleleft}!$.

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- Should model API calls, hence it also has the ports $out_i{}^{\triangleleft}!$.
- Real structure uses public-key cryptography to provide confidentiality and authenticity.
  - Message M from i to j encoded as $\mathcal{E}_j(\mathsf{sig}_i(M))$.
- Consists of PIOA-s $M_1, \ldots, M_n$. M_i has ports $in_i?$ and $out_i!$.
- M_i has ports $net_i^{\rightarrow}!$, $net_i^{\rightarrow\triangleleft}!$ and $net_i^{\leftarrow}?$ for (insecure) networking.
- Public keys are distributed over authentic channels.
  - M_i has ports $aut_{i,j}^{\rightarrow}!$, $aut_{i,j}^{\mathrm{a}}!$ and $aut_{j,i}^{\mathrm{a}}?$ for authentically communicating with party M_j.
  - M_i always writes identical messages to $aut_{i,j}^{\rightarrow}!$ and $aut_{i,j}^{\mathrm{a}}!$.

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  - M_i always writes identical messages to $aut_{i,j}^{\rightarrow}!$ and $aut_{i,j}^{\mathrm{a}}!$.
- $\mathsf{S} = \{in_1!, \ldots, in_n!, in_1{}^{\triangleleft}!, \ldots, in_n{}^{\triangleleft}!, out_1?, \ldots, out_n?\}$.

$\mathcal{I}$ is way too ideal

- Sending a message without initialization.
  - generating keys and distributing the public keys.
- Sending messages without delays. Guaranteed transmission.
- Traffic analysis.
- Concealing the length of messages.
- Transmitting only a number of messages polynomial to η.

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To simplify the presentation, we'll also

- Allow reordering and repetition of messages from one party to another.

The state of the PIOA $\mathcal{J}$

- Boolean $init_i$ — "has M_i generated the keys?"
- Boolean $init_{i,j}$ — "has M_j received the public keys of M_i?"
- Sequence of bit-strings $D_{i,j}$ — the messages party i has sent to party j.
- ℓ_i — the total length of messages party i has sent so far.

Initial values — `false`, ε, or 0.

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To set these values, $\mathcal{J}$ has to communicate with the adversary, too. It has the ports $adv^{\rightarrow}!$, $adv^{\rightarrow\triangleleft}!$ and $adv^{\leftarrow}?$ for that.

The transition function $\delta^{\mathfrak{I}}$

- On input (init) from $in_i?$: Set $init_i$ to $\texttt{true}$, write (init, i) to $adv^{\rightarrow}!$ and raise $adv^{\rightarrow\triangleleft}!$.
- On input (init, i, j) from $adv^{\leftarrow}?$: Set $init_{i,j}$ to $init_i$.
- On input (send, j, M) from $in_i?$: Do nothing if one of the following holds:
  - $|M| + \ell_i > p(\eta)$ for a fixed polynomial p;
  - $init_i \wedge init_{j,i} = \texttt{false}$.

  Otherwise add $|M|$ to ℓ_i and append M to $D_{i,j}$. Write $(\text{sent}, i, j, |M|)$ to $adv^{\rightarrow}!$ and raise $adv^{\rightarrow\triangleleft}!$.
- On input (recv, i, j, x) from $adv^{\leftarrow}?$: Do nothing if one of the following holds:
  - $init_j \wedge init_{i,j} = \texttt{false}$;
  - $x \leq 0$ or $|D_{i,j}| < x$.

  Otherwise write $(\text{received}, i, D_{i,j}[x])$ to $out_j!$ and raise $out_j{}^{\triangleleft}!$.

The state of the PIOA M_i

- The decryption key K_i^{d} and signing key K_i^{s}.
- The encryption keys K_j^{e} and verification keys K_j^{v} of all parties j.
- The length ℓ_i of the messages sent so far.

To operate, we have to fix

- IND-CCA-secure public key encryption system;
- EF-CMA-secure signature scheme.

The transition function δ^{M_i}

- On input (init) from $in_i?$: Generate keys $(K_i^{\mathrm{e}}, K_i^{\mathrm{d}})$ and $(K_i^{\mathrm{v}}, K_i^{\mathrm{s}})$. Ignore further (init)-requests. Write $(K_i^{\mathrm{e}}, K_i^{\mathrm{v}})$ to ports $aut_{i,j}^{\rightarrow}!$ and $aut_{i,j}^{\mathrm{a}}!$.
- On input $(k^{\mathrm{e}}, k^{\mathrm{v}})$ from $aut_{j,i}^{\mathrm{a}}?$: Initialize K_j^{e} and K_j^{v}.
- On input (send, j, M) from $in_i?$: If $|M| + \ell_i \leq p(\eta)$ and $K_i^{\mathrm{s}}, K_j^{\mathrm{e}}$ are defined
  - Let $v \leftarrow \mathcal{E}_{K_j^{\mathrm{e}}}(\mathsf{sig}_{K_i^{\mathrm{s}}}(i, j, M))$.
  - Add $|M|$ to ℓ_i.
  - Write (sent, j, v) to $net_i^{\rightarrow}!$ and raise $net_i^{\rightarrow\triangleleft}!$.
- On input (recv, j, v) from $net_i^{\leftarrow}?$: If the necessary keys are initialized and decryption and verification succeed (giving message M) then write $(\mathsf{received}, j, M)$ to $out_i!$ and raise $out_i^{\triangleleft}!$.

Ideal and real at a glance

$\mathcal{I}$:

$\underline{(\text{init}) \text{ from user } i\text{:}}$
set $init_i$, notify adversary.

$\underline{(\text{init}, i, j) \text{ from adversary:}}$
set $init_{i,j}$, if $init_i$ set.

$\underline{(\text{send}, j, M) \text{ from user } i\text{:}}$
store M in the sequence $D_{i,j}$;
send $(\text{send}, i, j, |M|)$ to adversary.
(only if $init_i \wedge init_{i,j}$)

$\underline{(\text{recv}, i, j, x) \text{ from adversary:}}$
send $(j, D_{i,j}[x])$ to user j.
(only if $init_j \wedge init_{j,i}$)

M_i:

$\underline{(\text{init}) \text{ from user:}}$
generate keys, send to adversary and others.

$\underline{(k^{\text{e}}, k^{\text{v}}) \text{ from } aut^{\text{a}}_{j,i}?\text{:}}$
set the public keys of j-th party

$\underline{(\text{send}, j, M) \text{ from user:}}$
Send j and $c = \mathcal{E}_{K^{\text{e}}_j}(\text{sig}_{K^{\text{s}}_i}(i, j, M))$ to the adversary
(only if K^{e}_j and K^{s}_i present)

$\underline{(j, c) \text{ from adversary:}}$
decrypt with K^{d}_i, check signature with K^{v}_j, send plaintext to user if OK
(only if K^{v}_j and K^{d}_i present)

The simulator

- The simulator translates between the ideal structure $\mathcal{I}$ and the "real" adversary.
- It has the following ports:
  - $adv^{\rightarrow}?$, $adv^{\leftarrow}!$, $adv^{\leftarrow\triangleleft}!$ for communicating with $\mathcal{I}$.
  - $net_i^{\rightarrow}!$, $net_i^{\rightarrow\triangleleft}!$, $net_i^{\leftarrow}?$, $aut_{i,j}^{\rightarrow}!$, $aut_{i,j}^{\mathrm{a}}!$, $aut_{j,i}^{\mathrm{a}}?$ for communicating with the "real" adversary.
    - Both ends of the channel $aut_{i,j}^{\mathrm{a}}$ are at Sim.
    - But the adversary schedules this channel.

Exercise. Construct the simulator.

Bisimulations

- A state-transition system is a tuple $(S, A, B, \to)$, where
  - S and A are the sets of states, input actions and output actions.
  - $\to$ is a partial function from $S \times A$ to $S \times B$.
    - Write $s \stackrel{a/b}{\to} t$ for $\to (s, a) = (t, b)$.
- An equivalence relation $\mathcal{R}$ over S is a bisimulation, if for all s, s', a, such that $s \mathrel{\mathcal{R}} s'$:
  - If $s \stackrel{a/b}{\to} t$ then exists $t' \in S$, such that $s' \stackrel{a/b}{\to} t'$ and $t \mathrel{\mathcal{R}} t'$.
- Two systems $(S, A, B, \to)$ and $(T, A, B, \Rightarrow)$ with starting states $s_0 \in S$ and $t_0 \in T$ are bisimilar, if there exists a bisimulation of $(S \mathbin{\dot{\cup}} T, A, B, \to \cup \Rightarrow)$ that relates s_0 and t_0.

Probabilistic bisimulations

- Let $(S, A, B, \rightarrow)$ be a probabilistic state-transition system. I.e.
  - S, A and B are the sets of states and input/output transitions.
  - $\rightarrow$ is a partial function from $S \times A$ to $\mathcal{D}(S \times B)$ (probability distributions over S).
- An equivalence relation $\mathcal{R}$ over S is a probabilistic bisimulation if $s \,\mathcal{R}\, s'$ implies
  - for each $a \in A$, $s \xrightarrow{a} D$ implies that there exists D', such that $s' \xrightarrow{a} D'$, and
  - for each $t \in S$ and $b \in B$: $\sum_{t' \in t/\mathcal{R}} D(t', b) = \sum_{t' \in t/\mathcal{R}} D'(t', b)$.
- Two probabilistic transition systems $(S, A, B, \rightarrow)$ and $(T, A, B, \Rightarrow)$ with starting states s_0 and t_0 are bisimilar if there exists a probabilistic bisimulation $\mathcal{R}$ of $(S \,\dot{\cup}\, T, A, \rightarrow \cup \Rightarrow)$ that relates s_0 and t_0.

Probabilistic bisimilarity

Bisimilarity of systems $(S, A, B, \to)$ and $(T, A, B, \Rightarrow)$ with starting states s_0, t_0 means that

- The sets S and T can be partitioned into $S_1 \dot\cup \cdots \dot\cup S_k$ and $T_1 \dot\cup \cdots \dot\cup T_k$, such that
  - ...some of S_i, T_i may be empty;
  - ...define a relation $\mathcal{R} \subseteq S \times T$, such that $s \,\mathcal{R}\, t$ iff $s \in S_i, t \in T_i$ for some i.
- For all $s \in S_i$, $t \in T_i$, $a \in A$:
- If $s \xrightarrow{a} D$ then $t \overset{a}{\Rightarrow} E$. Also, for each j and b: $\sum_{s' \in S_j} D(s', b) = \sum_{t' \in T_j} E(t', b)$.
- $s_0 \,\mathcal{R}\, t_0$.

Composition

Let the structures $(C_1, \mathsf{S}_1), \ldots, (C_k, \mathsf{S}_k)$ be given. We say that (C, S) is the composition of those structures if

- $C_1, \ldots, C_k$ are pairwise disjunct;
- the sets of ports of $C_1, \ldots, C_k$ are pairwise disjunct;
- $C = C_1 \cup \cdots \cup C_k$;
- $\mathsf{freeports}(C_i) \backslash \mathsf{S}_i \subseteq \mathsf{freeports}(C) \backslash S$ for all i.

Write $(C, \mathsf{S}) = (C_1, \mathsf{S}_1) \times \cdots \times (C_k, \mathsf{S}_k)$.

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Write $(C, \mathsf{S}) = (C_1, \mathsf{S}_1) \times \cdots \times (C_k, \mathsf{S}_k)$.

Theorem. Let

- $(C, \mathsf{S}) = (C_1, \mathsf{S}_1) \times (C_0, \mathsf{S}_0)$ and $(C', \mathsf{S}) = (C_1, \mathsf{S}_1) \times (C_0', \mathsf{S}_0)$;
- $(C_0, \mathsf{S}_0) \geq (C_0', \mathsf{S}_0')$.

Then $(C, \mathsf{S}) \geq (C', \mathsf{S})$.

Proof on the blackboard.

Power of composition

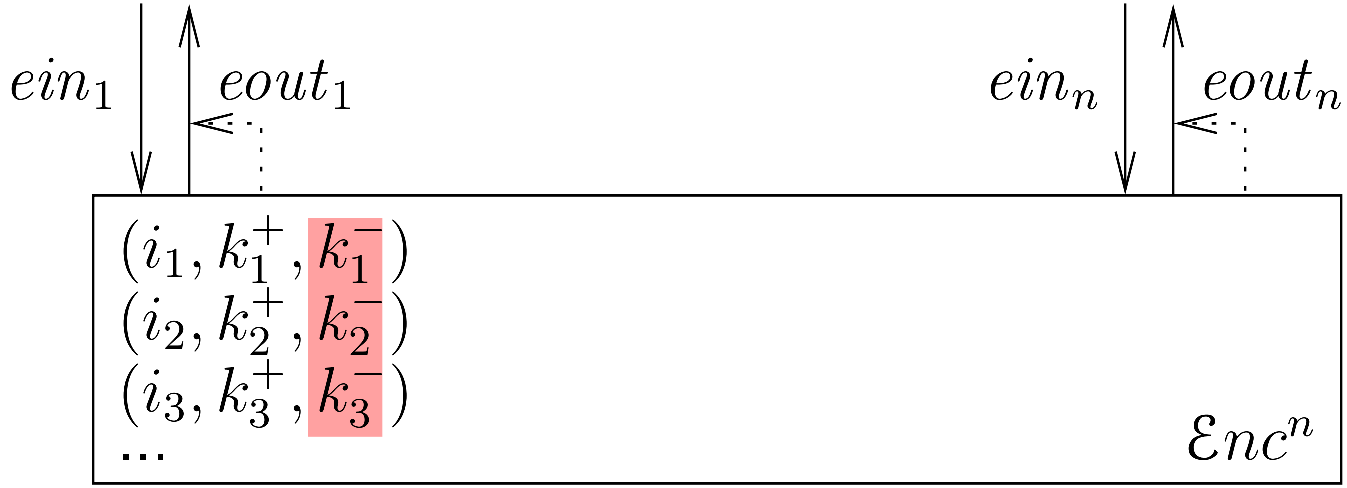
- The composition theorem gives the model its usefulness.
- One can construct a large system as follows:
  - Design it from the functionalities that have already been constructed.
    - add some glue code, if necessary.
  - Prove that it satisfies the needed (security) properties.
    - Assume the ideal implementations of existing functionalities.
  - Implement the system.
    - Use the real implementations of existing functionalities.
- The proofs of properties will hold for the real system.

Simulation for secure messaging

1. Separate encryption; replace it with an ideal encryption machine.
   - Same for signatures.
2. Define a probabilistic bisimulation with error sets between the states of $M_1 \| \cdots \| M_n$ and $\mathcal{I} \| Sim$.
3. Show that error sets have negligible probability.
   - The errors correspond to forging a signature or generating the same random value twice.
   - The first case may also be handled by defining a separate signature machine.
   - The second case may also be handled by defining the ideal machines in the appropriate way.

The PIOA $\mathcal{E}nc^n$

- Has ports $ein_i?$, $eout_i!$, $eout_i{}^{\triangleleft}!$ for $1 \leq i \leq n$.
- The machine M_i will get ports $ein_i!$, $ein_i{}^{\triangleleft}!$, $eout_i?$.
- On input (gen) from $ein_i?$: generate a new keypair (k^+, k^-), store (i, k^+, k^-), write k^+ to $eout_i!$, clock.
- On input (enc, k^+, M) from $ein_i?$: if k^+ has been stored as a public key, then compute $v \leftarrow \mathcal{E}(k^+, M)$, write v to $eout_i!$, clock.
- On input (dec, k^+, M) from $ein_i?$: if (i, k^+, k^-) has been stored, write $\mathcal{D}(k^-, M)$ to $eout_i!$, clock.

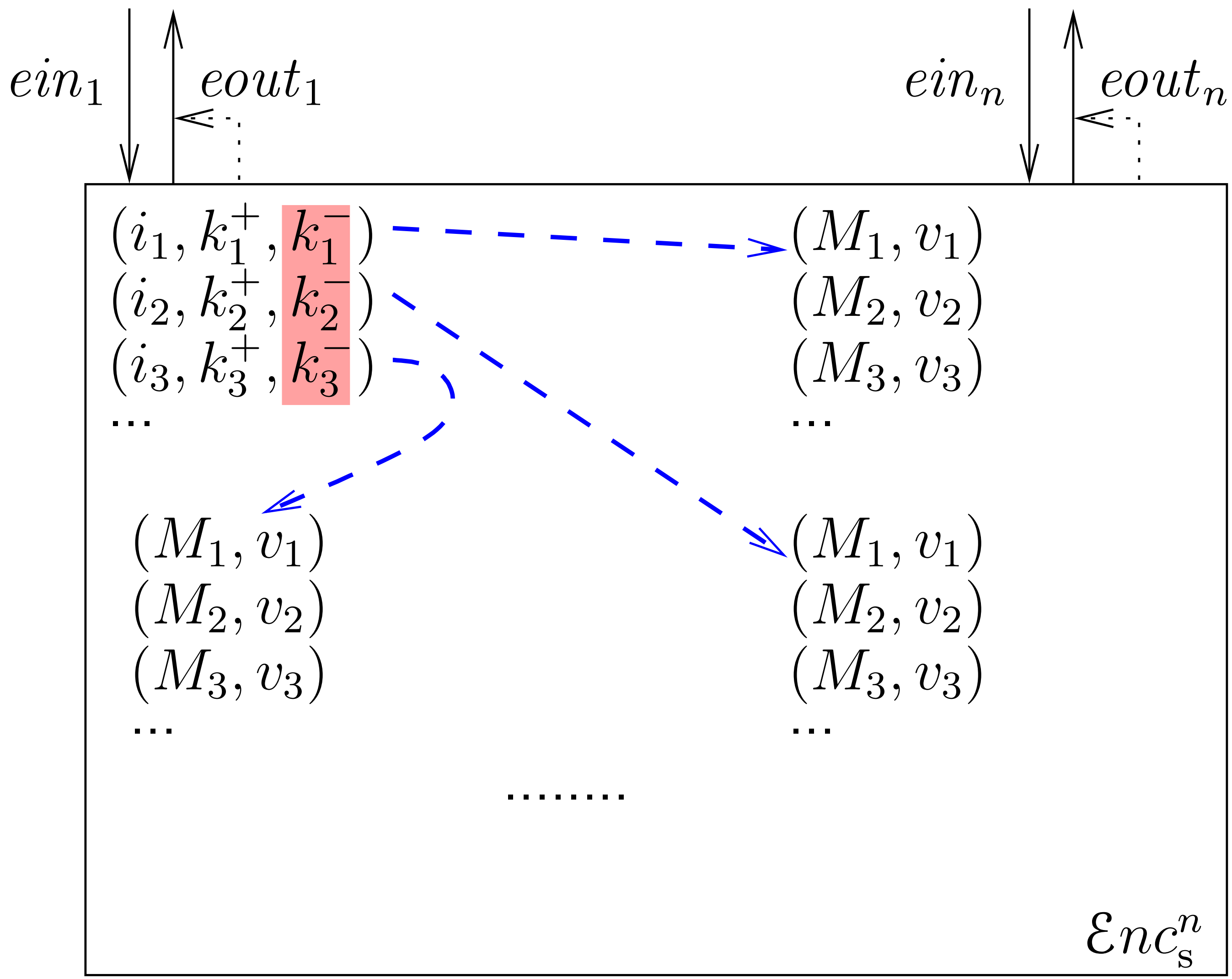


The PIOA $\mathcal{E}nc_{\mathrm{s}}^{n}$

- Has ports $ein_i?$, $eout_i!$, $eout_i{}^{\triangleleft}!$ for $1 \leq i \leq n$.
- The machine M_i will get ports $ein_i!$, $ein_i{}^{\triangleleft}!$, $eout_i?$.
- On input (gen) from $ein_i?$: generate a new keypair (k^+, k^-), store (i, k^+, k^-), write k^+ to $eout_i!$, clock.
- On input (enc, k^+, M) from $ein_i?$: if k^+ has been stored as a public key, then compute $v \leftarrow \mathcal{E}(k^+, 0^{|M|})$, store (k^+, M, v), write v to $eout_i!$, clock.
  - Recompute v until it differs from all previous v-s.
- On input (dec, k^+, v) from $ein_i?$: if (i, k^+, k^-) has been stored, then
  - if (k^+, M, v) has been stored for some v, then write v to $eout_i!$, clock.
  - otherwise write $\mathcal{D}(k^-, M)$ to $eout_i!$, clock.

$\mathcal{E}nc^n \geq \mathcal{E}nc_{\mathrm{s}}^{n}$ (black-box). **Exercise.** Describe the simulator.

The PIOA $\mathcal{E}nc^n_{\mathrm{s}}$



The PIOA $\mathcal{S}ig^n$

- Has ports $sin_i?$, $sout_i!$, $sout_i^{\triangleleft}!$ for $1 \leq i \leq n$.
- The machine M_i will get necessary ports for using $\mathcal{S}ig^n$ as by API calls.
- On input (gen) from $sin_i?$: generate a new keypair (k^+, k^-), store (i, k^+, k^-), write k^+ to $sout_i!$, clock.
- On input (sig, k^+, M) from $sin_i?$: if (i, k^+, k^-) has been stored then compute $v \leftarrow \text{sig}(k^-, M)$, write v to $sout_i!$, clock.
- On input (ver, k^+, s) from $sin_i?$: if k^+ has been stored then write $\text{ver}(k^+, s)$ to $sout_i!$, clock.

The PIOA $\mathcal{S}ig^n_{\mathrm{s}}$

- Has ports $sin_i?$, $sout_i!$, $sout_i{}^{\triangleleft}!$ for $1 \leq i \leq n$.
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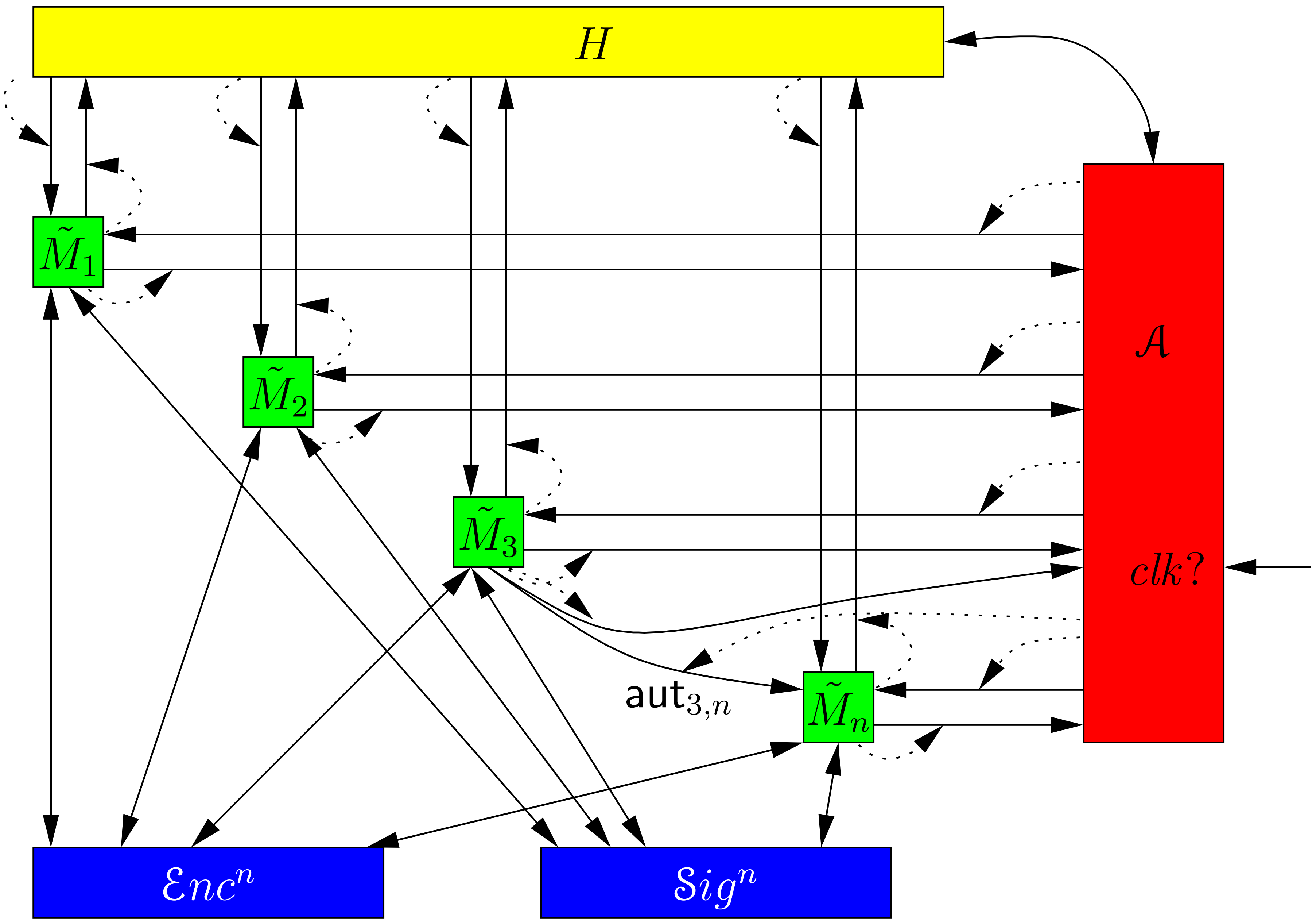
- On input (sig, k^+, M) from $sin_i?$: if (i, k^+, k^-) has been stored then compute $v \leftarrow \mathsf{sig}(k^-, M)$, store (k^+, M), write v to $sout_i!$, clock.
- On input (ver, k^+, s) from $sin_i?$: if k^+ has been stored then write $\mathsf{ver}(k^+, s) \wedge$ "(k^+, M) has been stored" to $sout_i!$, clock.

Theorem. $\mathcal{S}ig^n \geq \mathcal{S}ig^n_{\mathrm{s}}$.

Modified real structure

- Instead of generating the encryption keys, and encrypting and decrypting themselves, machines M_i query the machine $\mathcal{E}nc^n$.
- We can then replace $\mathcal{E}nc^n$ with $\mathcal{E}nc^n_{\mathrm{s}}$. The original structure was at least as secure as the modified structure.
- Same for signatures. . .
- Denote the modified machines by $\tilde{M}_i$.

This is at least as secure as. . .



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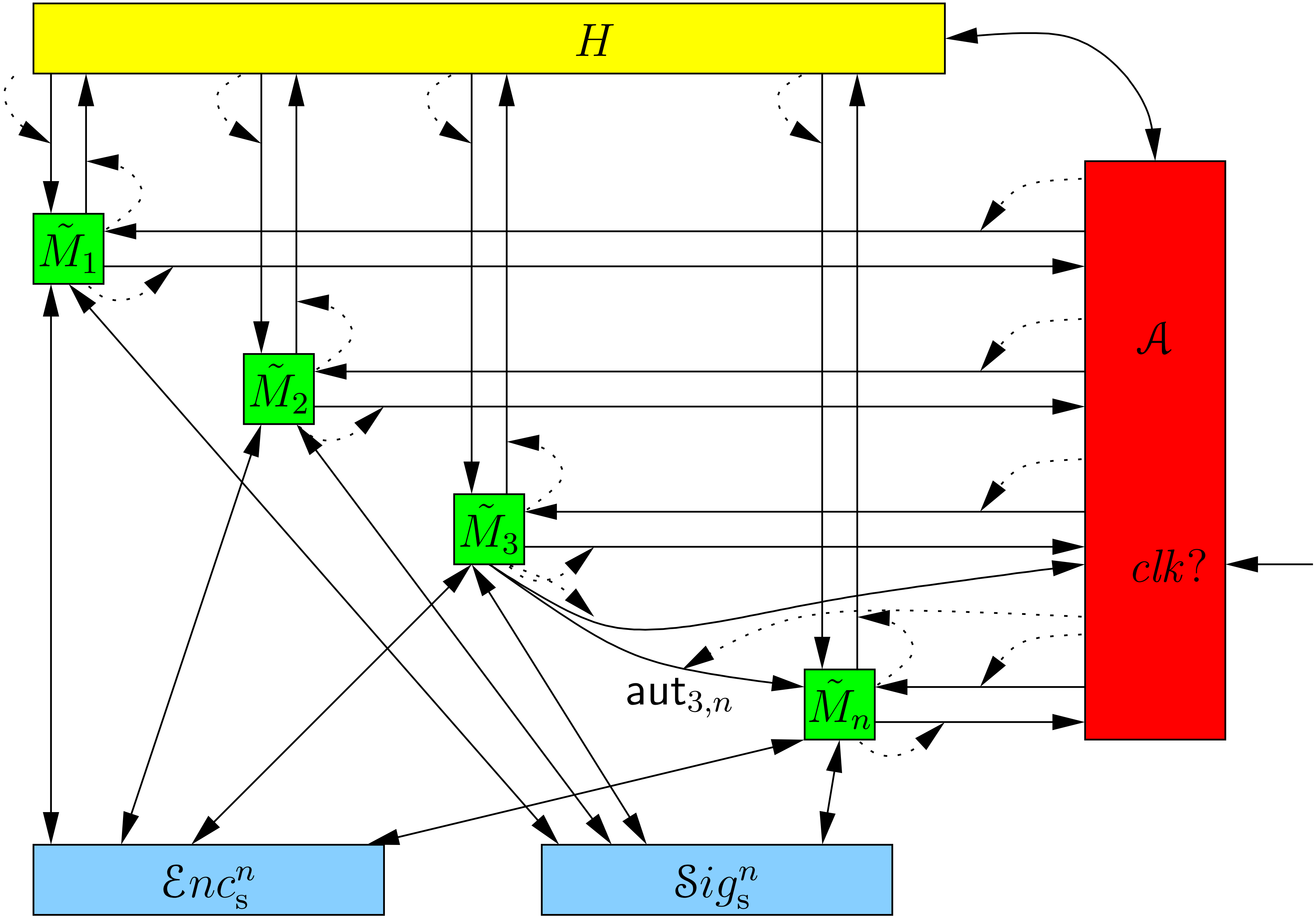
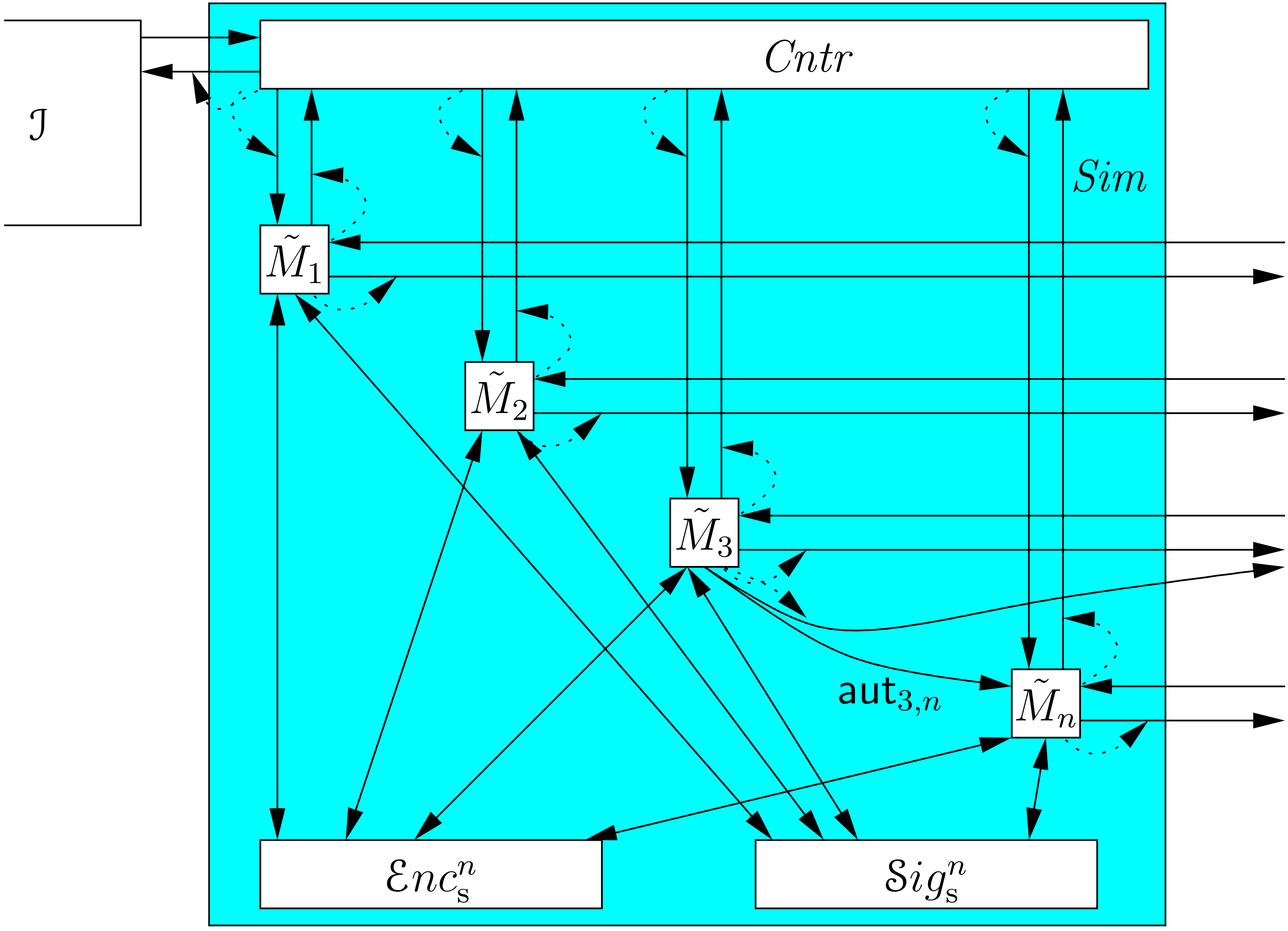


H
$\tilde{M}_1$
$\tilde{M}_2$
$\tilde{M}_3$
$\tilde{M}_n$
$\mathcal{A}$
$clk?$
$\mathsf{aut}_{3,n}$
$\mathcal{E}nc_{\mathrm{s}}^n$
$\mathcal{S}ig_{\mathrm{s}}^n$

The state of the real structure

- State of $\tilde{M}_i$ — the keys K^{e}_j and K^{v}_j $(1 \leq j \leq n)$.
  - If some K is defined at several machines, then they are equal.
- State of $\mathcal{E}nc^n_{\mathrm{s}}$:
  - key triples (i, k^+, k^-), where k^+ is the same as K^{e}_i.
  - text triples (k^+, M, v), where k^+ also occurs in a key triple.
- State of $\mathcal{S}ig^n_{\mathrm{s}}$:
  - key triples (i, k^+, k^-), where k^+ is the same as K^{v}_i.
  - text pairs (k^+, M), where k^+ also occurs in a key triple.
- Possibly (during initialization) the keys in the buffers of the channels $aut^{\mathrm{a}}_{i,j}$.
- No messages are in the buffers of newly introduced channels ein_i etc.
- The buffers of channels connected to H or A are not part of the state.

The simulator Sim



The simulator Sim

- Consists of the real structure and one extra machine $Cntr$. Its state contains message sequences D'_{ij} for all $1 \leq i, j \leq n$.
- The ports $in_i?$, $out_i!$, $out_i{}^{\triangleleft}!$ of $\tilde{M}_i$ are renamed to $cin_i?$, $cout_i!$, $cout_i{}^{\triangleleft}!$.
- Machine $Cntr$ has ports $cin_i!$, $cin_i{}^{\triangleleft}!$, $cout_i?$, $adv^{\leftarrow}!$, $adv^{\leftarrow\triangleleft}!$, $adv^{\rightarrow}?$.
- On input (init, i) from $adv^{\rightarrow}?$ write (init) to $cin_i!$ and clock it.
- On input $(k^{\mathrm{e}}, k^{\mathrm{v}})$ from $aut^{\mathrm{a}}_{j,i}?$: the machine $\tilde{M}_i$ additionally writes $(\mathsf{recvkeys}, j)$ to $cout_i!$ and clocks it.
- Receiving $(\mathsf{recvkeys}, j)$ from $cout_i?$, machine $Cntr$ writes (init, j, i) to $adv^{\leftarrow}!$ and clocks it.
- Receiving (send, i, j, l) from $adv^{\rightarrow}?$, the machine $Cntr$ generates a new message M of length l, appends it to $D'_{i,j}$, writes (send, j, M) to $cin_i!$, clocks it.
- Receiving $(\mathsf{received}, i, M)$ from $cout_j?$, the machine $Cntr$ finds x, such that $D'_{i,j}[x] = M$, writes (recv, i, j, x) to $adv^{\leftarrow}!$, clocks it.

The state of $\mathcal{J} \| Sim$

- The state of real structure. Additionally
- For each i, j, the sequences $D'_{i,j}$ of messages that the machine $Cntr$ has generated.
- Initialization bits $init_i$, $init_{i,j}$.
- The sequences of messages $D_{i,j}$ that party i has sent to party j. (stored in $\mathcal{J}$)

The state of $\mathcal{I}\|Sim$

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- For each i, j, the sequences $D'_{i,j}$ of messages that the machine $Cntr$ has generated.
- Initialization bits $init_i$, $init_{i,j}$.
- The sequences of messages $D_{i,j}$ that party i has sent to party j. (stored in $\mathcal{I}$)

Lemma. If $\mathcal{I}\|Sim$ is not currently running, then

- $|D_{ij}| = |D'_{i,j}|$ and the lengths of the messages in the sequences $D_{i,j}$ and $D'_{i,j}$ are pairwise equal.
- If $init_i$ then $\tilde{M}_i$ has requested the generation of keys. If $init_{i,j}$ then $\tilde{M}_j$ has received the keys of $\tilde{M}_i$. The opposite also holds.
- The signed messages in $\mathcal{S}ig^n_{\mathrm{s}}$ are exactly of the form (i, j, M) where M is in the sequence $D'_{i,j}$. The encrypted messages in $\mathcal{E}nc^n_{\mathrm{s}}$ are exactly those signed messages.

Bisimilarity for secure channels

Relating the states of real and (ideal$\|$simulator) structures:

- The states of $\tilde{M}_i$, $\mathcal{E}nc_{\mathrm{s}}^n$, $\mathcal{S}ig_{\mathrm{s}}^n$ must be equal.
- The rest of the state of $\mathcal{I}\|Sim$ must satisfy the lemma we had above.

The relationship must hold only if either H or A is currently running.

- Now consider all possible inputs that the real structure or (ideal$\|$simulator) may receive. Show that they react to it in the identical manner.

Extension: static corruptions

- Allow the adversary to corrupt the parties before the start of the run (before party has received the (init)-command).
- In the real functionality: machine M_i may accept a command (corrupt) from the port $net_i^{\leftarrow}?$.
- It forwards all messages it receives directly to the adversary (over the channel $net_i^{\rightarrow}$) and receives from the adversary the messages it has to write to other ports.

Exercise. How should we change the ideal functionality? The simulator?
Exercise. Why is it hard to model dynamic corruptions?

Home exercise

Present a simulatable functionality for secure channels (not allowing corruptions) that preserves the order of messages and does not allow their duplication.

Please use the defined secure messaging functionality as a building block (use the composition).

Deadline: Mid-January.

An UC voting functionality

Let there be m voters and n talliers. Let the possible votes be in $\{0, \ldots, L-1\}$.
All voters will give their votes. All authorities agree on the result. The adversary will not learn individual votes.

- At the voting phase, the voters write their encrypted votes to a bulletin board.
  - Use threshold homomorphic encryption.
  - Talliers have the shares of the secret key.
- Everybody can see the encrypted votes and combine them to the encryption of the tally.
- After the voting period, the talliers publish the plaintext shares of the tally.
- Everybody can combine those shares and learn the voting result.

The ideal functionality

- The ideal functionality $\mathcal{I}_{\text{VOTE}}$ has the standard ports... $in_i^V?$, $out_i^V!$, $out_i^{V\triangleleft}!$, $in_i^T?$, $out_i^T!$, $out_i^{T\triangleleft}!$, $adv^{\leftarrow}?$, $adv^{\rightarrow}!$, $adv^{\rightarrow\triangleleft}!$.
- First expect (init, sid)-command from the adversary.

The ideal functionality

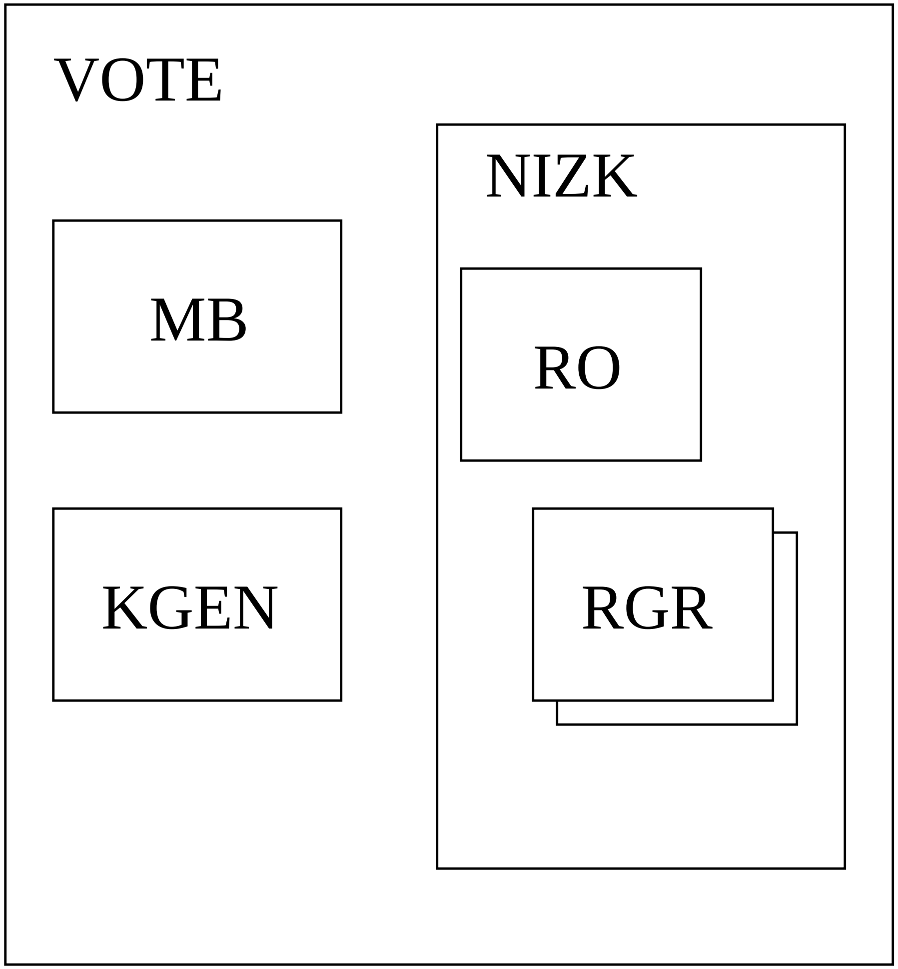
- The ideal functionality $\mathcal{I}_{\text{VOTE}}$ has the standard ports... $in_i^V?$, $out_i^V!$, $out_i^{V\triangleleft}!$, $in_i^T?$, $out_i^T!$, $out_i^{T\triangleleft}!$, $adv^{\leftarrow}?$, $adv^{\rightarrow}!$, $adv^{\rightarrow\triangleleft}!$.
- First expect (init, sid)-command from the adversary.
- On input (vote, sid, v) from V_i store $(\text{vote}, sid, V_i, v, 0)$, send (vote, sid, V_i) to the adversary, ignore further votes from V_i in session sid.
- On input $(\text{accept}, sid, V_i)$ from the adversary, change the flag from 0 to 1 in $(\text{vote}, sid, V_i, v, \_)$.

The ideal functionality

- The ideal functionality $\mathcal{I}_{\text{VOTE}}$ has the standard ports... $in_i^V?$, $out_i^V!$, $out_i^{V\triangleleft}!$, $in_i^T?$, $out_i^T!$, $out_i^{T\triangleleft}!$, $adv^{\leftarrow}?$, $adv^{\rightarrow}!$, $adv^{\rightarrow\triangleleft}!$.
- First expect (init, sid)-command from the adversary.
- On input (vote, sid, v) from V_i store $(\text{vote}, sid, V_i, v, 0)$, send (vote, sid, V_i) to the adversary, ignore further votes from V_i in session sid.
- On input $(\text{accept}, sid, V_i)$ from the adversary, change the flag from 0 to 1 in $(\text{vote}, sid, V_i, v, \_)$.
- On input (result, sid) from the adversary, add up the votes in session sid with flag 1, store (result, sid, r) and send it to the adversary.
- On input $(\text{giveresult}, sid, i)$ from the adversary send (result, sid, r) to voter V_i or tallier T_{i-m}.

Building blocks

- Message board
  - Synchronous communication
- Homomorphic threshold encryption
  - MPC (for key generation)
- NIZK proofs
  - Random oracle
  - Generation of random elements of a group



Message board

Ideal functionality $\mathcal{I}_{\text{MB}}$ for parties $P_1, \ldots, P_n$ is the following:

- On input (bcast, sid, v) from P_i, store $(\text{bcast}, i, sid, v)$. Accept no further $(\text{bcast}, sid, \ldots)$-queries from P_i. Send $(\text{bcast}, sid, i, v)$ to the adversary.
- On input (pass, sid, i) from the adversary, if $(\text{bcast}, i, sid, v)$ has been stored, store (post, sid, i, v).
- On input (tally, sid) from the adversary, accept no more $(\text{bcast}, sid, \ldots)$ and $(\text{pass}, sid, \ldots)$-requests.
- On input $(\text{request}, sid)$ from P_j, if (tally, sid) has been received before, send all stored $(\text{post}, sid, \ldots)$-tuples to P_j (as a single message).

Realization requires reliable channels or smth.

ZK proofs

The ideal functionality $\mathcal{I}_{\text{ZK}}$ for parties $P_1, \ldots, P_n$ and witnessing relation $\mathcal{R}$ is the following

- On input $(\text{prove}, sid, P_j, x, w)$ from a party P_i:
  - Check that $(x, w) \in \mathcal{R}$;
  - Store (P_i, P_j, sid, x);
  - Send $(\text{prove}, P_i, P_j, sid, x)$ to the adversary.
  - Accept no more $(\text{prove}, sid, \ldots)$ queries from P_i.
- On input $(\text{proofok}, P_i, P_j, sid, x)$ from the adversary send $(\text{proof}, sid, P_i, x)$ to P_j.

NIZK proofs

The ideal functionality $\mathcal{F}_{\text{NIZK}}$ for parties $P_1, \ldots, P_n$ and witnessing relation $\mathcal{R}$ is the following

- On input $(\text{prove}, sid, x, w)$ from a party P_i:
  - Check that $(x, w) \in \mathcal{R}$;
  - Send (proof, sid, x) to the adversary.
  - Accept no more $(\text{prove}, sid, \ldots)$ queries from P_i.
  - Wait for a query of the form $(\text{proof}, sid, x, \pi)$ from the adversary.
    - A restriction on the adversary.
    - Can be justified for the ideal functionalities.
    - This topic warrants a deeper research.
  - Store (sid, x, π).
  - Send $(\text{proof}, sid, x, \pi)$ to P_i.

NIZK proofs

- On input $(\text{prove}, sid, x, w, \pi)$ from the adversary:
  - Check that $(x, w) \in \mathcal{R}$;
  - Store (sid, x, π).
- On input $(\text{verify}, sid, x, \pi)$ from P_j check whether (sid, x, π) is stored. If it is then
  - Return $(\text{verifyok}, sid, x)$.

  If it is not then
  - Send $(\text{witness?}, sid, x)$ to the adversary.
  - Wait for a query of the form $(\text{prove}, sid, x, w, \pi)$ from the adversary.
  - Handle $(\text{prove}, sid, x, w, \pi)$ as before.
  - If $(x, w) \in \mathcal{R}$ then return $(\text{verifyok}, sid, x)$ to P_j.

Random oracles

The random oracle functionality $\mathcal{J}_{\mathrm{RO}}$ for n parties is the following:

- On input x by any party or the adversary
  - If (x, r) is already stored for some r, return r.
  - Otherwise generate $r \in_R \{0,1\}^{p(\eta)}$, store (x, r) and return r.

  $\mathcal{J}_{\mathrm{RO}}$ works as a subroutine.

Generating a random element of a group

Let G be a fixed group (depends on η only), with a prime cardinality and hard DDH problem. The functionality $\mathcal{I}_{\mathrm{RGR}}$ is the following:

- On input (init) by the adversary generates a random element of G and returns it to the adversary.
- On input (init, i) marks that it may answer to party P_i.
- On input (get) from a party returns the generated element, if allowed.

Realization:

- The machines M_i are initialized by the adversary.
- M_i generates a random element $g_i \in G$, secret shares it;
- The shared values are multiplied and the result is opened.
- A (get) by a party allows it to learn the computed value.
- Uses secure channels functionality.

Exercise. How to simulate?

Protocol realizing NIZK

- Idea: on input $(\text{prove}, sid, x, w)$ from party P_i the machine M_i commits to w and outputs x, $C(w)$, and a NIZK proof that $C(w)$ is hiding a witness for x.
- Initialization: parties get two random elements $g, h \in G$ using two copies of $\mathcal{J}_{\text{RGR}}$.
  - Ignore user's query if (get) to $\mathcal{J}_{\text{RGR}}$-s gets no response.
- Let us use the following commitment scheme (G is a group with cardinality $\#G$ and hard DDH problem):
  - To commit to $m \in G$, generate a random $r \in \{0, \ldots \#G - 1\}$. The commitment is $(g^r, m \cdot h^r)$.
  - The opening of the previous commitment is r.

Exercise. How to verify? What is this commitment scheme? What can be said about its security?

Protocol realizing NIZK

- There exists a ZK protocol for proving that a commitment c hides a witness w, such that $(x, w) \in \mathcal{R}$.
- For honest verifiers, this protocol has three rounds — commitment (or witness), challenge and response.
  - It depends on $\mathcal{R}$ (and the commitment scheme).
  - Let $A(x, C(w), w, r)$ generate the witness and $Z(x, C(w), w, r, a, c)$ compute the response.
  - Challenge is a random string. Let $\mathcal{V}(x, C(w), a, c, z)$ be the verification algorithm at the end.
- The whole proof π for (x, sid) consists of
  - $C(w)$, a random string $\bar{r}$;
  - $a \leftarrow A(x, C(w), w, r)$;
  - $z \leftarrow Z(x, C(w), w, r, a, H(x, a, sid, \bar{r}))$
- $(\text{proof}, sid, x, \pi)$ is sent back to the user.

Protocol realizing NIZK

- On input $(\text{verify}, sid, x, \pi)$ from the user, machine M_j verifies that proof:
  - Computes $c = H(x, a, sid, \bar{r})$ (by invoking $\mathcal{F}_{\mathrm{RO}}$) and verifies $\mathcal{V}(x, C, a, c, z)$.

  If correct, responds with $(\text{verifyok}, sid, x)$.

Simulation

The simulator communicates with

- the ideal functionality: possible commands are
  - $(\mathsf{proof}, i, sid, x)$;
  - $(\mathsf{witness?}, sid, x, \pi)$.
- the real adversary: possible commands are
  - (init) and (init, i) for two copies of $\mathcal{I}_{\mathrm{RGR}}$;
  - queries to the random oracle $\mathcal{I}_{\mathrm{RO}}$.
    - Answer the queries to $\mathcal{I}_{\mathrm{RO}}$ in the normal way.

Simulator: initialization

On the very first invocation:

- Generate random elements $g, h \in G$.

On (init) and (init, i) from the adversary for functionalities $\mathcal{I}_{\text{RGR}}$:

- Record that these commands have been received.

Simulating $(\mathsf{proof}, i, sid, x)$

- The query $(\text{prove}, sid, x, w)$ was made by party P_i to $\mathcal{I}_{\text{NIZK}}$.
- Where do we get w?

Simulating $(\mathsf{proof}, i, sid, x)$

- The query $(\mathsf{prove}, sid, x, w)$ was made by party P_i to $\mathcal{I}_{\text{NIZK}}$.
- Where do we get w? **We don't get it at all.**
- Let C be the commitment of a random element w';
- **Simulate** the ZK proof of $(x, w') \in \mathcal{R}$:
  - Let c be a random challenge.
  - Let (a, z) be suitable witness and response for showing that C is the commitment of a suitable witness of x in $\mathcal{R}$.
- Let $\bar{r}$ be a random string, such that $(x, a, sid, \bar{r})$ has not been a query to $\mathcal{I}_{\text{RO}}$.

Simulating $(\mathsf{proof}, i, sid, x)$

- The query $(\mathsf{prove}, sid, x, w)$ was made by party P_i to $\mathcal{J}_{\mathrm{NIZK}}$.
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  - Let (a, z) be suitable witness and response for showing that C is the commitment of a suitable witness of x in $\mathcal{R}$.
- Let $\bar{r}$ be a random string, such that $(x, a, sid, \bar{r})$ has not been a query to $\mathcal{J}_{\mathrm{RO}}$.
- **Define** $H(x, a, sid, \bar{r}) := c$. Let $\pi = (C, \bar{r}, a, z)$.
- Send $(\mathsf{proof}, sid, x, i, \pi)$ to $\mathcal{J}_{\mathrm{NIZK}}$.

(*Programmable* random oracle)

Simulating $(\text{witness?}, sid, x, \pi)$

This is called if the real adversary has independently constructed a valid proof.

- Change the simulator as follows:
  - Initialization: the simulator generates g and h so, that it knows $\log_g h$.
- On a $(\text{witness?}, \ldots)$-query, the simulator checks whether the proof $\pi = (C, \bar{r}, a, z)$ is correct.
- If it is, then it extracts the witness w from C by ElGamal decryption.

- After that, it sends $(\text{prove}, sid, x, w, \pi)$ to $\mathcal{I}_{\text{NIZK}}$.

Exercise. What if C does not contain a valid witness?

Corruptions

- The real adversary may send (corrupt)-command to some machine M_i.
  - Static corruptions — only at the beginning.
  - Adaptive corruptions — any time.
- The machine responds with its current state.
- Afterwards, M_i "becomes a part of" the adversary.
  - Forwards all received messages to the adversary.
  - M_i accesses other components on behalf of the adversary.
  - No more traffic between M_i and the user.
- Possibility to corrupt players has to be taken into account when specifying ideal functionalities.
  - The ideal adversary may send $(\mathrm{corrupt}, i)$ to the functionality.
    - The simulator will make these queries if the real adversary corrupted someone.
  - The functionality may change the handling of the i-th party.

Corruptions and functionalities

- Random oracles — impossible to corrupt.
- Generating a random element of the group:
  - Implementations uses MPC techniques.
  - Tolerates adaptive corruptions of less than $n/3$ participants.
  - If party i is corrupted, then $\mathcal{I}_{\mathrm{RGR}}$
    - Gives no output to the i-th party.
    - Forwards to the adversary all requests from the i-th party.
  - If too many parties are corrupted (at least $n/3$) then $\mathcal{I}_{\mathrm{RGR}}$ gives all control to the adversary.
  - The simulator simply acts as a forwarder between a corrupted party and the adversary.

Corrupting $\mathcal{J}_{\text{NIZK}}$

- The realization of NIZK uses $\mathcal{J}_{\text{RGR}}$.
  - It fails if there are at least $n/3$ corrupt parties.
- It has no other weaknesses.

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  - It fails if there are at least $n/3$ corrupt parties.
- It has no other weaknesses.
- If party i is corrupted in $\mathcal{J}_{\mathrm{NIZK}}$ then it stops talking to the user.
  - The adversary may prove things on user's behalf.
- If at least $n/3$ parties are corrupted then $\mathcal{J}_{\mathrm{NIZK}}$ gives up.

Corrupting $\mathcal{J}_{\mathrm{NIZK}}$

- The realization of NIZK uses $\mathcal{J}_{\mathrm{RGR}}$.
  - It fails if there are at least $n/3$ corrupt parties.
- It has no other weaknesses.
- If party i is corrupted in $\mathcal{J}_{\mathrm{NIZK}}$ then it stops talking to the user.
  - The adversary may prove things on user's behalf.
- If at least $n/3$ parties are corrupted then $\mathcal{J}_{\mathrm{NIZK}}$ gives up.
- The simulator corrupts i-th party of $\mathcal{J}_{\mathrm{NIZK}}$ if M_i is corrupted or the i-th party in $\mathcal{J}_{\mathrm{RGR}}$ is corrupted.

Exercise

How should corruptions be integrated to $\mathcal{I}_{\text{MB}}$?

Ideal functionality $\mathcal{I}_{\text{MB}}$ for parties $P_1, \ldots, P_n$ is the following:

- On input (bcast, sid, v) from P_i, store $(\text{bcast}, i, sid, v)$. Accept no further $(\text{bcast}, sid, \ldots)$-queries from P_i. Send $(\text{bcast}, sid, i, v)$ to the adversary.
- On input (pass, sid, i) from the adversary, if $(\text{bcast}, i, sid, v)$ has been stored, store (post, sid, i, v).
- On input (tally, sid) from the adversary, accept no more $(\text{bcast}, sid, \ldots)$ and $(\text{pass}, sid, \ldots)$-requests.
- On input $(\text{request}, sid, i)$ from P_j, if (tally, sid) has been received before, send all stored $(\text{post}, sid, \ldots)$-tuples to P_j (as a single message).

Homomorphic encryption

- A public-key encryption system $(\mathcal{K}, \mathcal{E}, \mathcal{D})$.
- The set of plaintexts is a ring.
- There is an operation $\oplus$ on ciphertexts, such that if $\mathcal{D}(k^-, c_1) = v_1$ and $\mathcal{D}(k^-, c_2) = v_2$ then $\mathcal{D}(k^-, c_1 \oplus c_2) = v_1 + v_2$.
- Security — IND-CPA.

Homomorphic encryption

- A public-key encryption system $(\mathcal{K}, \mathcal{E}, \mathcal{D})$.
- The set of plaintexts is a ring.
- There is an operation $\oplus$ on ciphertexts, such that if $\mathcal{D}(k^-, c_1) = v_1$ and $\mathcal{D}(k^-, c_2) = v_2$ then $\mathcal{D}(k^-, c_1 \oplus c_2) = v_1 + v_2$.
- Security — IND-CPA.
- In a threshold encryption system, the secret key is shared. There are shares $k_1^-, \ldots, k_n^-$.
- Also, there are public verification keys $k_1^{\mathrm{v}}, \ldots, k_n^{\mathrm{v}}$ that are used to verify that the authorities have correctly computed the shares of the plaintext.
  - . . . like in verifiable secret sharing.
- We use secure MPC to generate $k^+, k_1^-, \ldots, k_n^-, k_1^{\mathrm{v}}, \ldots, k_n^{\mathrm{v}}$.
  - This can be modeled by an ideal functionality $\mathcal{I}_{\mathrm{KGEN}}$.
  - There are more efficient means of generation than general MPC.

Key generation

The ideal functionality $\mathcal{I}_{\mathrm{KGEN}}$ for m users and n authorities works as follows:

- On input $(\text{generate}, sid)$ from the adversary, generates new keys. and gives the keys $k^+, k_1^{\mathrm{v}}, \ldots, k_n^{\mathrm{v}}$ to the adversary.
- On input $(\text{getkeys}, sid)$ from a party, gives the party this party's generated keys. (works like subroutine)
- Breaks down if there are at least $(m+n)/3$ corrupt parties.

Each voting session needs new keys, otherwise chosen-ciphertext attacks are possible.

Voting protocol

- Voter machines $M_1^V, \ldots, M_m^V$, tallier machines $M_1^T, \ldots, M_n^T$.
- The first time some M_i^V or M_i^T is activated, it asks for its key(s) from $\mathcal{T}_{\text{KGEN}}$ and receives them.
- On input (vote, sid, v) from the user the machine M_i^V
  - Let $c_i \leftarrow \mathcal{E}_{k^+}(\text{Encode}(v))$. Make a NIZK proof π_i that c_i contains a correct vote. Send $(\text{bcast}, sid \| 0, (c_i, \pi_i))$ to $\mathcal{T}_{\text{MB}}$.
- On input (count, sid) from the adversary the machine M_i^T
  - Sends $(\text{request}, sid \| 0, i)$ to $\mathcal{T}_{\text{MB}}$ and receives all the votes and correctness proofs $(c_1, \pi_1), \ldots, (c_m, \pi_m)$.
  - Checks the validity of the proofs, using $\mathcal{T}_{\text{NIZK}}$.
  - Multiplies the valid votes and decrypts the result, using k_i^-. Let the result of the decryption be d_i. Makes a NIZK proof ξ_i that d_i is a valid decryption and sends $(\text{bcast}, sid \| 1, (d_i, \xi_i))$ to $\mathcal{T}_{\text{MB}}$.
    - The proof also uses k_i^{v}.

Voting protocol

- On input (result, sid) from the adversary any machine
  - Sends $(\text{request}, sid\|0, i)$ to $\mathcal{I}_{\text{MB}}$ and receives all the votes and correctness proofs $(c_1, \pi_1), \ldots, (c_m, \pi_m)$.
  - Checks the validity of the proofs, using $\mathcal{I}_{\text{NIZK}}$.
  - Multiplies the valid votes, let the result be c.
  - Sends $(\text{request}, sid\|1, i)$ to $\mathcal{I}_{\text{MB}}$ and receives the shares of the result $d_1, \ldots, d_n$ together with proofs $\xi_1, \ldots, \xi_n$.
  - Check the validity of those proofs.
  - Combines a number of valid shares to form the final result r.
  - Sends (result, sid, r) to the user.

Exercise. What kind of corruptions are tolerated here?

The simulator — interface

The simulator encapsulates $\mathcal{F}_{\text{MB}}$, $\mathcal{F}_{\text{NIZK}}$, $\mathcal{F}_{\text{KGEN}}$.
The simulator handles the following commands:

- From $\mathcal{F}_{\text{VOTE}}$:
  - (vote, sid, i) — V_i has voted (but don't know, how).
  - (result, sid, r) — the result of the voting session sid.
- From the real adversary:
  - (count, sid) for M_i^T — produce the share of the voting result.
  - (result, sid) for any M — combine the shares of the result and send it to the user.
  - Corruptions; messages on behalf of corrupted parties.

The simulator — interface

- From the real adversary (on behalf of $\mathcal{F}_{\text{MB}}$):
  - $(\textsf{pass}, sid, i)$ — lets the message sent by M_i to pass.
  - $(\textsf{tally}, sid)$ — finishes round sid.
  - $(\textsf{bcast}, sid, i, v)$ — broadcast by a corrupt party.
- From the real adversary (on behalf of $\mathcal{F}_{\text{NIZK}}$):
  - $(\textsf{proof}, sid, x, \pi)$ — generate a proof token π for an honest prover.
  - $(\textsf{prove}, sid, x, w, \pi)$ — the adversary proves something himself.
- From the real adversary (on behalf of $\mathcal{F}_{\text{KGEN}}$):
  - $(\textsf{generate}, sid)$ — generates the keys.

The simulator — interface

The simulator issues the following commands:

To $\mathcal{I}_{\text{VOTE}}$:

(init, sid)
(accept, sid, i)
(result, sid)
(giveresult, sid, i)
(corrupt, i)
(vote, sid, i, v)

To the real adversary (as $\mathcal{I}_{\text{MB}}$):

(bcast, sid, i, v)

To the real adversary (as $\mathcal{I}_{\text{NIZK}}$):

(proof, i, sid, x)
(witness?, sid, x, π)

To the real adversary (as $\mathcal{I}_{\text{KGEN}}$):

(keys, $sid, k^+, k_1^{\text{v}}, \ldots, k_n^{\text{v}}$)

The simulator — initialization

- On the first activation with a new sid:
  - Generates keys $k^+, k_1^-, \ldots, k_n^-, k_1^{\mathrm{v}}, \ldots, k_n^{\mathrm{v}}$ for this session.
- When receiving $(\text{generate}, sid)$ from the adversary for $\mathcal{I}_{\text{KGEN}}$,
  - marks that voting can now commence;
  - sends (init, sid) to $\mathcal{I}_{\text{VOTE}}$.
- Corruptions by the adversary are forwarded to $\mathcal{I}_{\text{VOTE}}$ and recorded.

The simulator — voting

- On input (vote, sid, i) from $\mathcal{J}_{\text{VOTE}}$:
  - Let the encrypted vote be $c \leftarrow \mathcal{E}_{k^+}(0)$.
  - Make a NIZK proof π that this vote is valid.
    - Going to $\mathcal{J}_{\text{NIZK}}$'s waiting state, as necessary.
  - Broadcast (using $\mathcal{J}_{\text{MB}}$) the pair (c, π) on behalf of voter i.
- On input (pass, sid, i), if the vote was broadcast for the voter P_i:
  - Send (accept, sid, i) back to $\mathcal{J}_{\text{VOTE}}$.
- If a corrupt party i puts a vote to the message board and makes a valid proof for it:
  - Decrypt that vote. Let its value be v.
  - Send (vote, sid, i, v) to $\mathcal{J}_{\text{VOTE}}$.

The simulator — tallying

On input $(\text{tally}, sid\|0)$ from the adversary for $\mathcal{I}_{\text{MB}}$:

- Close the voting session sid, accept counting queries.
- Send (result, sid) to $\mathcal{I}_{\text{VOTE}}$.
- Get the voting result r from $\mathcal{I}_{\text{VOTE}}$ and store it.

The simulator — counting

On input (count, sid) from the adversary for the tallier T_i:

- Check the proofs of all votes (c_i, π_i) using $\mathcal{F}_{\text{NIZK}}$.
  - Going to wait-state, if necessary.
- Let C be the product of all votes with valid proofs.
- For talliers $T_1, \ldots, T_n$, let $d_1, \ldots, d_n$ be
  - if T_i is corrupt, then $d_i = \mathcal{D}(k_i^-, C)$;
  - if T_i is honest, then a d_i is simulated value

  such that $d_1, \ldots, d_n$ combine to r.
  - $d_1, \ldots, d_n$ are generated at the first (count, sid)-query.
- Make a NIZK proof ξ_i for the share d_i.
- Broadcast (d_i, ξ_i) in session $sid \| 1$ using $\mathcal{F}_{\text{MB}}$.
- A corrupt tallier can broadcast anything. But only (d_i, ξ_i) for the valid d_i is accepted at the next step.

The simulator — reporting the results

On input (result, sid) from the adversary for any voter or tallier i:

- Takes all votes (c_j, π_j) and all shares of the result (d_j, ξ_j).
- Verifies all correctness proofs of votes.
- Multiplies the valid votes.
- Verifies the correctness proofs of shares.
- If sufficiently many proofs are correct then sends $(\text{giveresult}, sid, i)$ to $\mathcal{I}_{\text{VOTE}}$.

Damgård-Jurik encryption system

- A homomorphic threshold encryption system
- Somewhat RSA-like
  - Operations are modulo n^s, where n is a RSA modulus.
  - Easy to recover i from $(1+n)^i \bmod n^s$.
- Maybe in the lecture...
- Otherwise see http://www.daimi.au.dk/~ivan/GenPaillier_finaljour.ps

Secure MPC from thresh. homom. encr.

Computationally secure against malicious coalitions with size less than the threshold.

- Function given as a circuit with multiplications and additions.
- The value on each wire is represented as its encryption, known to all.

- Addition gate — everybody can add encrypted values by themselves.
- Multiplication of a and b (encryptions are $\overline{a}$ and $\overline{b}$):
  - Each party P_i chooses a random d_i, broadcasts $\overline{d_i}$, proves in ZK that it knows d_i.
  - Let $d = d_1 + \cdots + d_n$. Then $\overline{d} = \overline{d_1} \oplus \cdots \oplus \overline{d_n}$.
  - Decrypt $\overline{a} \oplus \overline{d} = \overline{a + d}$, let everybody know it.
  - Let $\overline{a_1} = \overline{a + d} \ominus \overline{d_1}$ and $\overline{a_i} = \ominus \overline{d_i}$. P_i knows a_i.
  - P_i broadcasts $a_i \odot \overline{b} = \overline{a_i b}$ and proves in ZK that he computed it correctly.
  - Everybody computes $\overline{a_1 b} \oplus \cdots \oplus \overline{a_n b} = \overline{ab}$.