Primer on finitary probability theory

Events and probabilities

- A random experiment is any activity whose outcome is not uniquely determined by the preconditions.
  - Encompasses all things happening that we are interested in.
- An elementary event is a possible outcome of the random experiment.
  - The sample space Ω is the set of all elementary events.
  - We assume Ω is finite or countable.
- An event is a subset of Ω.
- The probability is a function $\mathrm{pr} : \Omega \rightarrow [0, 1]$, s.t. $\sum_{\omega\in\Omega} \mathrm{pr}(\omega) = 1$.
  - Should be thought as a fixed function. There aren't several different probabilities.
- The probability of an event $A \subseteq \Omega$ is $\Pr(A) = \sum_{\omega\in A} \mathrm{pr}(\omega)$.

Some elementary theorems

- $\Pr(\emptyset) = 0$
- $\Pr(\Omega) = 1$
- $\Pr(\Omega \backslash A) = 1 - \Pr(A)$
- If $|\Omega| = n$ and $\mathrm{pr}(\omega) = 1/n$ for all $\omega \in \Omega$ then $\Pr(A) = |A|/n$
  - In this case, finding $\Pr(A)$ reduces to finding $|A|$.
- If $A \cap B = \emptyset$ then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
  - Also holds for a countable number of mutually exclusive events
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
  - If $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$ then A and B are independent.

Conditional probability

- Let $\Pr(B) > 0$.
- Definition: $\Pr(A|B) = \Pr(A \cap B)/\Pr(B)$.
- Bayes' formula: $\Pr(A|B) \cdot \Pr(B) = \Pr(B|A) \cdot \Pr(A)$.
- $\Pr(\cdot|B)$ could also serve as a probability measure.
  - $\Pr(\cdot|B)$ satisfies the same theorems as $\Pr(\cdot)$.

Random variables

- A random variable is a function $\mathbf{X} : \Omega \rightarrow X$ for any set X.
- If P is a predicate on X then $\mathrm{P}(\mathbf{X})$ is a random event.
- Assume now $X = \mathbb{R}$.
- The average of $\mathbf{X}$ is $\mathbf{E}(\mathbf{X}) = \sum_{\omega \in \Omega} \mathrm{pr}(\omega) \cdot \mathbf{X}(\omega)$.
- Theorems:
    - $\mathbf{E}(\mathbf{X} + \mathbf{Y}) = \mathbf{E}(\mathbf{X}) + \mathbf{E}(\mathbf{Y})$. $\mathbf{E}(\lambda \cdot \mathbf{X}) = \lambda \cdot \mathbf{E}(\mathbf{X})$.
        - Even if $\mathbf{X}$ and $\mathbf{Y}$ are dependent.
    - If $\Pr(\mathbf{X} \leq \mathbf{Y}) = 1$ then $\mathbf{E}(\mathbf{X}) \leq \mathbf{E}(\mathbf{Y})$.
- Two random variables $\mathbf{X}$ and $\mathbf{Y}$ are independent if for all x and y, $\Pr(\mathbf{X} = x, \mathbf{Y} = y) = \Pr(\mathbf{X} = x) \cdot \Pr(\mathbf{Y} = y)$.
- If $\mathbf{X}$ and $\mathbf{Y}$ are independent then $\mathbf{E}(\mathbf{X} \cdot \mathbf{Y}) = \mathbf{E}(\mathbf{X}) \cdot \mathbf{E}(\mathbf{Y})$.